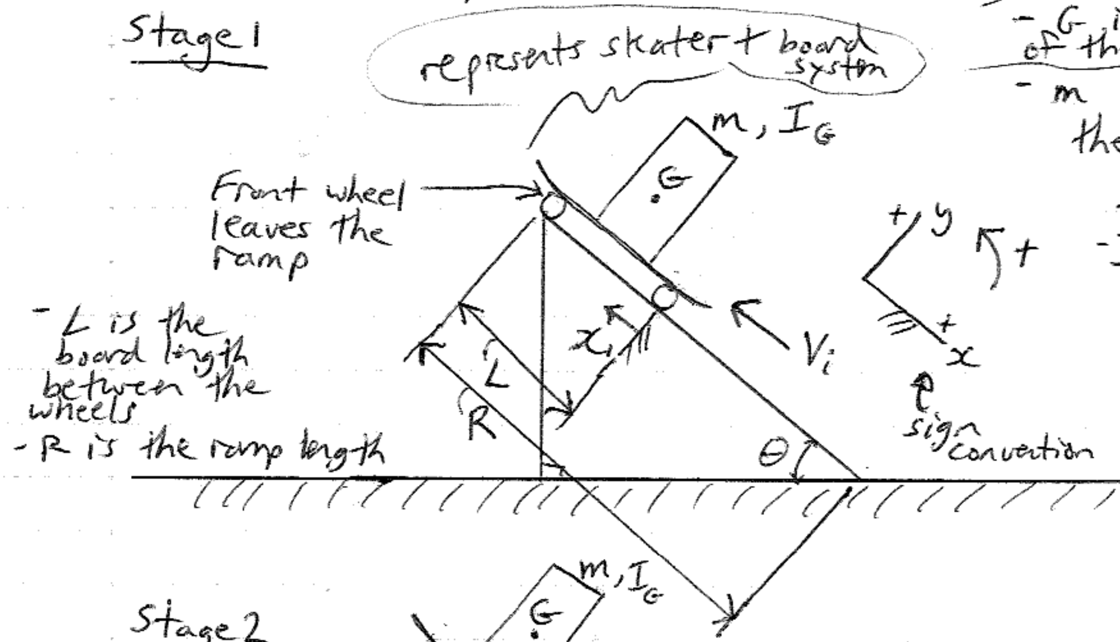
May 3, 2014

Does the length of a skateboard affect the jumping distance over a ramp?

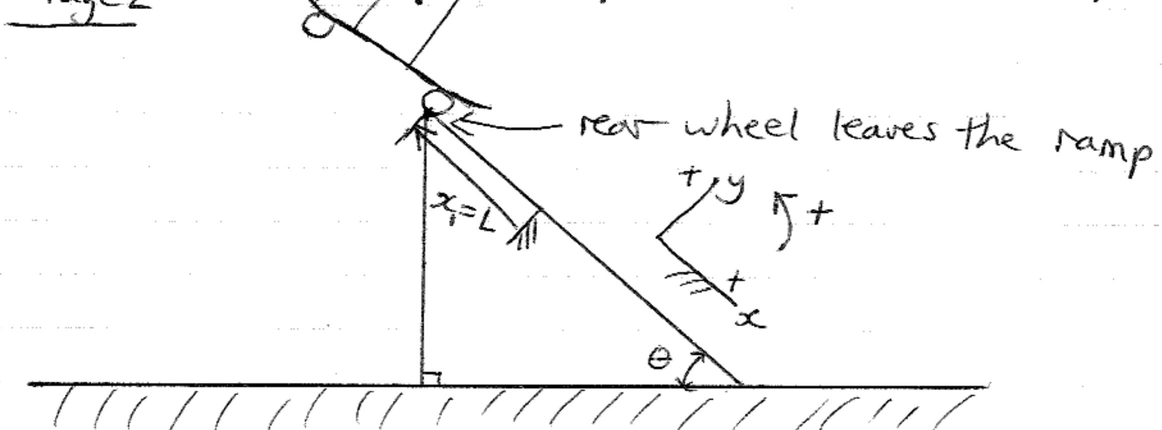
Solution: Break down the problem into three stages as shown below, with variables as shown.

Stage 1

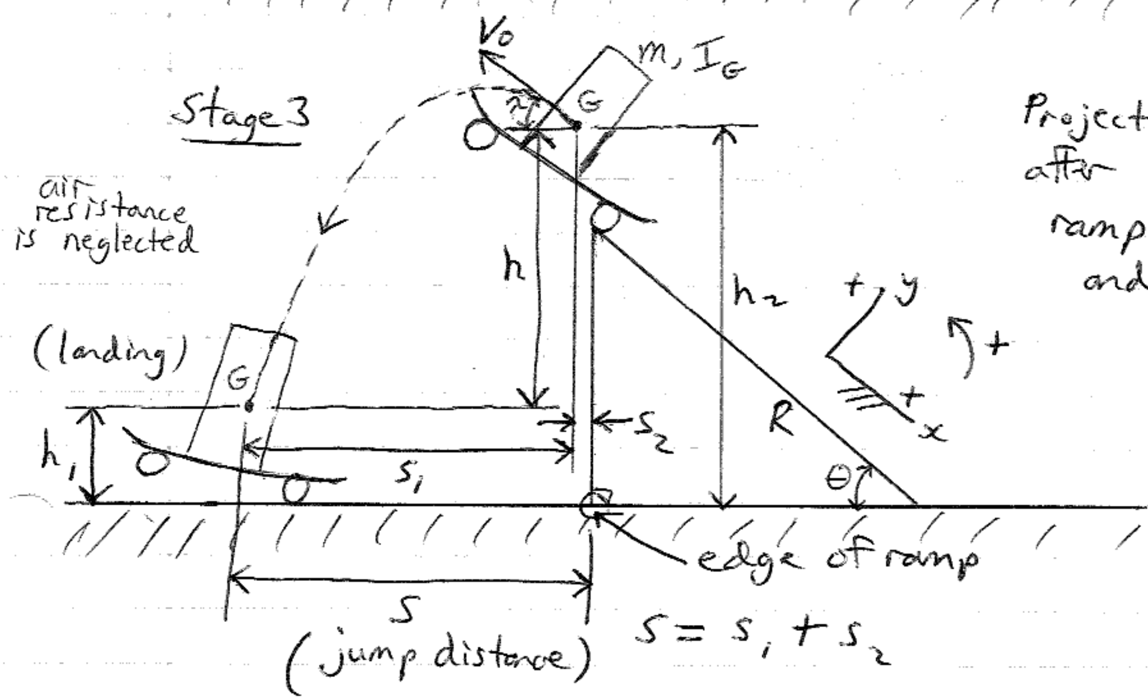


- G is the center of mass of the skater + board system
- m is the mass of the skater + board system
- I_G is the rotational inertia of the skater + board system, about G
- V_i is the board velocity just as the front wheel leaves the ramp

Stage 2



Stage 3



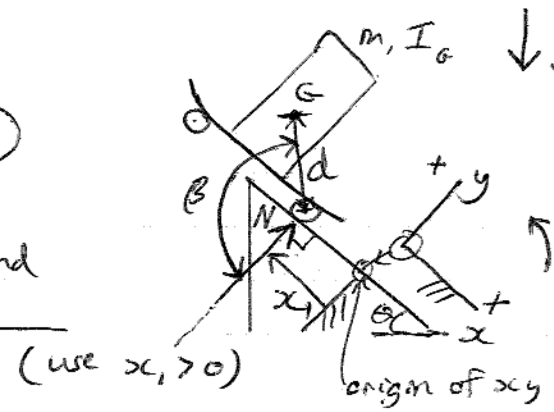
Projectile motion occurs after board leaves ramp (between stage 2 and the landing)

$$s = s_1 + s_2$$

Perform dynamics analysis between stage 1 and stage 2, and a projectile motion analysis for stage 3. This is a 2D problem.

Ⓘ

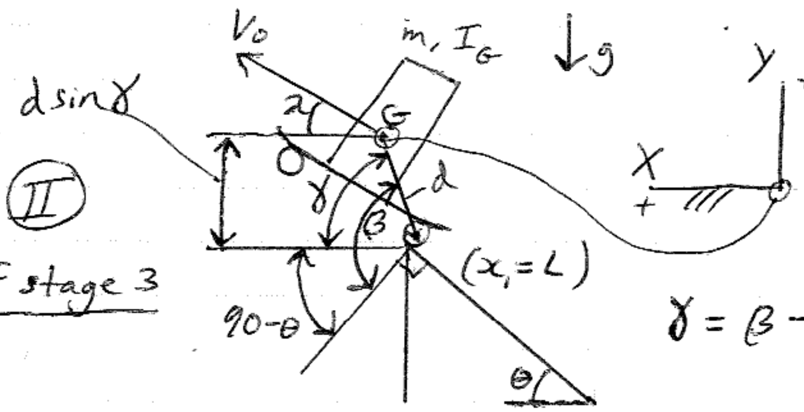
Between stage 1 and stage 2



- g is the acceleration due to gravity
- N is the normal force acting on the rear wheel. The force in the x-direction is negligible

Ⅱ

Beginning of stage 3



Put fixed XY coordinate system at G (at the beginning of stage 3)

$$\gamma = \beta - (90 - \theta) = \beta + \theta - 90$$
(at $x_1 = L$)

From Ⓘ we can do a dynamics analysis.

The equations of motion to consider are:

Newton's second law
① $\Sigma F_x = m a_{Gx}$
② $\Sigma F_y = m a_{Gy}$

Torque equation
③ $\Sigma M_G = I_G \alpha$

ΣM_G is the sum of the torque about the center of mass G

where ΣF_x, ΣF_y are the sum of the forces in the x, y directions, respectively
a_{Gx} is the acceleration of G in the x-direction
a_{Gy} is the acceleration of G in the y-direction
α is the angular acceleration of the skater + board system, treated as a rigid body

Now,

$$\Sigma F_x = mg\sin\theta$$
$$\Sigma F_y = N - mg\cos\theta$$
$$\Sigma M_G = N d \sin\beta$$

$\alpha = -\ddot{\beta}$ (as required by the sign convention)

$g = 9.8\ m/s^2$

relative to xy frame { To easily find a_{Gx} and a_{Gy}, determine first the position (x_G, y_G) and then differentiate twice with respect to time to determine the acceleration.

Now, $x_G = -x_1 - d\sin\beta$
$y_G = -d\cos\beta$

Differentiating twice and then simplifying we get

$$a_{Gx} = -\ddot{x}_1 - d\left(-\sin\beta\cdot(\dot{\beta})^2 + \cos\beta\cdot(\ddot{\beta})\right)$$

$$a_{Gy} = d\left(\cos\beta\cdot(\dot{\beta})^2 + \sin\beta\cdot(\ddot{\beta})\right)$$

Equations ① - ③ then become

① $g\sin\theta = -\ddot{x}_1 - d\left(-\sin\beta\cdot(\dot{\beta})^2 + \cos\beta\cdot(\ddot{\beta})\right)$

② $N - mg\cos\theta = m\left[d(\cos\beta\cdot(\dot{\beta})^2 + \sin\beta\cdot(\ddot{\beta}))\right]$

③ $I_G(-\ddot{\beta}) = Nd\sin\beta$

Equations ① - ③ are differential equations which can be solved numerically. The initial conditions are: $\beta(0) = \beta_0$, $\dot{\beta}(0) = 0$, $x_1(0) = 0$, $\dot{x}_1(0) = V_i$

These equation have to be solved over the range $0 \le x_1 \le L$. When $x_1 = L$ the board leaves the ramp and we have projectile motion.

The math is not shown here | V_i is affected by board length. This was accounted for using conservation of energy. If the board is shorter, or longer, by an amount ΔL, then point G is approximately $\frac{\Delta L}{2}\cdot\sin\theta$ higher or lower (respectively) in vertical direction.

From (II), $\beta = \beta(\text{at } x_1 = L)$

To find V_0 we must first determine $\dot{x}_G$ and $\dot{y}_G$.

$$\left.\begin{aligned} \dot{x}_G &= -\dot{x}_1 - d\cos\beta\cdot(\dot{\beta}) \\ \dot{y}_G &= d\sin\beta\cdot(\dot{\beta}) \end{aligned}\right\} \text{at } x_1 = L$$

Thus, $V_0 = \sqrt{(\dot{x}_G)^2 + (\dot{y}_G)^2}$ (velocity of G at $x_1 = L$)

Angle $\lambda = \operatorname{atan}\left(\dfrac{V_{GY}}{V_{GX}}\right)$

In XY frame,

$$V_{GX} = -\dot{x}_G\cos\theta - \dot{y}_G\sin\theta \quad (\text{for } \dot{x}_G, \dot{y}_G \text{ at } x_1 = L)$$

$$V_{GY} = -\dot{x}_G\sin\theta + \dot{y}_G\cos\theta$$

From stage 3, $h_2 = R\sin\theta + d\sin\gamma$

and $h = h_2 - \underbrace{h_1}_{\text{known}}$ and $s_2 = d\cos\gamma$ looking at (II)

Now, $s_1 = \dfrac{V_0^2\cos(\lambda)}{g}\left(\sin(\lambda) + \sqrt{\sin^2(\lambda) + \dfrac{2gh}{V_0^2}}\right)$

$g = 9.8\,m/s^2$

Projectile motion equation for horizontal distance along X-direction. The derivation of this equation is not shown here.

Therefore, $s = s_1 + s_2$ (jump distance for center of mass G)

In my simulation, using Excel, I determined that s is negligibly affected by board length (within the range of available board lengths).

However, if the jump distance is measured from the edge of the ramp to the back of the board then a shorter board results in longer jump distance, and a longer board will result in shorter jump distance. Therefore, in a jumping competition everyone should have the same length of board L.

An example of the variables used in the simulation are:

$m = 68\ Kg$ $\qquad g = 9.8\ m/s^2$
$I_G = 17\ Kg \cdot m^2$
$L = 0.5\ m$
$d = 1.25\ m$
$\theta = 30°$
$R = 3\ m$
$h_1 = 1.2\ m$
$\beta(0) = 2.9\ rads$
$\dot{\beta}(0) = 0$
$x_1(0) = 0$
$\dot{x}_1(0) = 4\ m/s = V_i$

In the calculations we have:

simulation results
$$\begin{cases} V_0 = 3.176\ m/s \\ \lambda = 0.338\ rads \\ h = 1.525\ m \\ \underline{s = 1.83\ m} \ (\text{jump distance}) \end{cases}$$