Physics Questions And Solutions Ebook

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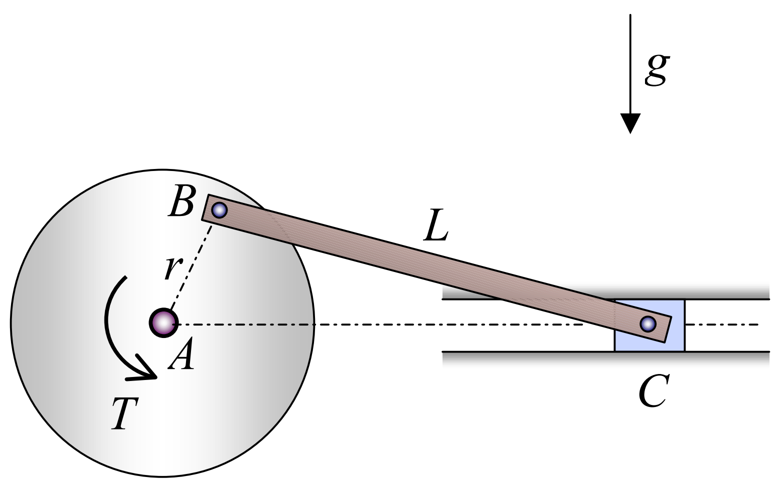
Problem # 1

A crank drive mechanism is illustrated below. A uniform linkage *BC* of length *L* connects a flywheel of radius *r* (rotating about fixed point *A*) to a piston at *C* that slides back and forth in a hollow shaft. A variable torque *T* is applied to the flywheel such that it rotates at a constant angular velocity. Show that for one full rotation of the flywheel, energy is conserved for the entire system; consisting of flywheel, linkage, and piston (assuming no friction).

Note that gravity *g* is acting downwards, as shown.

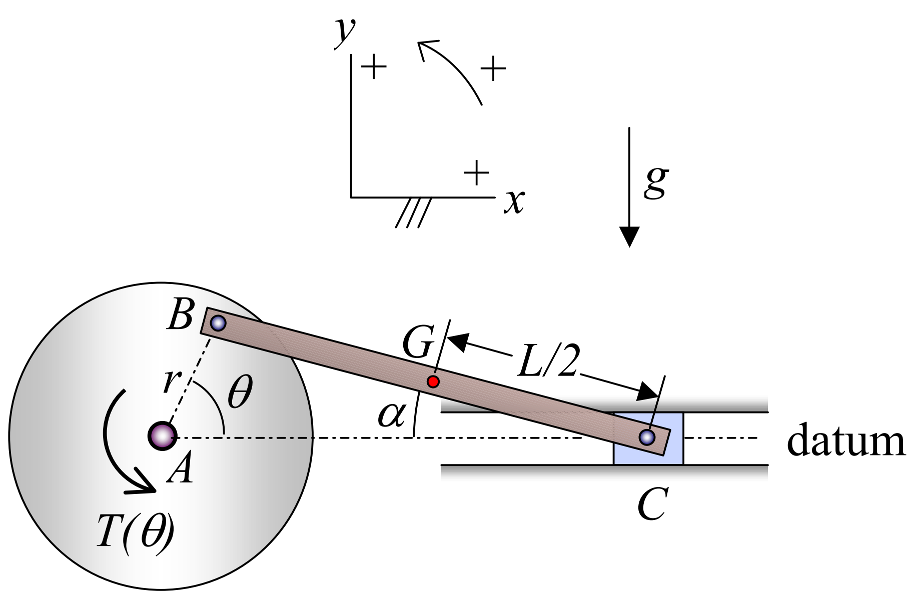
Even though energy is conserved for the system, why is it a good idea to make the components of the drive mechanism as light as possible (with the exception of the flywheel)?



Problem # 1 Solution

Assuming there is no friction anywhere, we wish to show that for one full rotation of the flywheel no net energy is added to the system.

To start, it is useful to redraw the figure and introduce some new variables, along with sign convention. These variables are chosen for convenience and to help simplify the solution.



where point *G* (red dot) is the center of mass of the uniform linkage *BC*.

The horizontal dashed line represents the datum (reference line) upon which gravitational potential energy is measured.

The system consists of flywheel, linkage, and piston. Using geometry we can relate the position of these components.

The change in total energy of the flywheel, linkage, and piston is equal to the energy added by the torque *T*.

Using Calculus we can set up an expression where the change in total energy of the system over a small angle $d\theta$ is equal to the change in total energy of the flywheel, linkage, and piston (over $d\theta$).

Now, find expressions for the total energy of the flywheel, linkage, and piston.

The total energy of the flywheel (kinetic and gravitational potential) is:

$$E_{flywheel} = \frac{1}{2} I_A \left(\frac{d\theta}{dt} \right)^2$$

where I_A is the rotational inertia of the flywheel about fixed point A (about an axis pointing out of the page), and $d\theta/dt$ is the angular velocity of the flywheel, in radians/second. This angular velocity is given as constant. The flywheel has zero gravitational potential energy since the datum passes through the center of mass of the flywheel, at point A, which is a fixed point.

The total energy of the linkage (kinetic and gravitational potential) is:

$$E_{linkage} = \frac{1}{2} I_G \left(\frac{d\alpha}{dt} \right)^2 + \frac{1}{2} m_{BC} v_G^2 + m_{BC} g \left(\frac{L}{2} \sin\alpha \right)$$

Where:

I_G is the rotational inertia of the linkage about the center of mass G (about an axis pointing out of the page)

$d\alpha/dt$ is the angular velocity of the linkage, in radians/second

m_{BC} is the mass of the linkage

v_G is the velocity of the center of mass G of the linkage

g is the acceleration due to gravity

The last term on the right of the above equation is the gravitational potential energy of the linkage (with line AC chosen as the datum). Since point G is changing in height, the gravitational potential energy of the linkage must be accounted for.

To find the velocity v_G determine the position of point G and then differentiate it with respect to time.

For convenience, define the origin of the xy axes at fixed point A.

The horizontal position of point *G* is given as:

$$x_G = r\cos\theta + \frac{L}{2}\cos\alpha$$

Differentiate x_G with respect to time. We get

$$\frac{dx_G}{dt} = -r\sin\theta \cdot \frac{d\theta}{dt} - \frac{L}{2}\sin\alpha \cdot \frac{d\alpha}{dt}$$

The vertical position of point *G* is given as:

$$y_G = \frac{L}{2}\sin\alpha$$

Differentiate y_G with respect to time. We get

$$\frac{dy_G}{dt} = \frac{L}{2}\cos\alpha \cdot \frac{d\alpha}{dt}$$

The velocity of the center of mass *G* of the linkage is therefore

$$v_G^2 = \left(\frac{dx_G}{dt}\right)^2 + \left(\frac{dy_G}{dt}\right)^2 = \left(r\sin\theta \cdot \frac{d\theta}{dt}\right)^2$$

$$+ rL\sin\theta \cdot \sin\alpha \cdot \frac{d\theta}{dt} \cdot \frac{d\alpha}{dt} + \left(\frac{L}{2} \cdot \frac{d\alpha}{dt}\right)^2$$

The total energy of the piston (kinetic and gravitational potential) is:

$$E_{piston} = \frac{1}{2} m_C v_C^2$$

where m_C is the mass of the piston and v_C is the velocity of the piston. The piston has zero gravitational potential energy since the datum passes through the center of mass of the piston, at every stage of its motion.

Over a small angle $d\theta$ the energy added to the system by the torque T is:

$$dE_{in} = Td\theta$$

where θ is measured in radians.

This must be equal to the change in total energy of the system (comprising of the flywheel, linkage, and piston).

The change in total energy of the system is therefore given as:

$$dE_{system} = d(E_{flywheel} + E_{linkage} + E_{piston}) \qquad (1)$$

So,

$$\begin{aligned} Td\theta &= d(E_{flywheel} + E_{linkage} + E_{piston}) \\ &= dE_{flywheel} + dE_{linkage} + dE_{piston} \end{aligned}$$

Since the flywheel is rotating at constant angular velocity its energy does not change. So,

$$dE_{flywheel} = 0$$

Therefore,

$$Td\theta = dE_{linkage} + dE_{piston}$$

For one full rotation we integrate this equation. Thus,

$$\int_{\theta_o}^{\theta_o+2\pi} T d\theta = \int_{\theta_o}^{\theta_o+2\pi} (dE_{linkage} + dE_{piston})$$

$$= \int_{\theta_o}^{\theta_o+2\pi} dE_{linkage} + \int_{\theta_o}^{\theta_o+2\pi} dE_{piston}$$

where θ_o is an arbitrary initial angle and 2π represents a full rotation, in radians (equal to 360°).

Now, the above equation becomes

$$\int_{\theta_o}^{\theta_o+2\pi} T d\theta = E_{linkage} \Big|_{\theta_o}^{\theta_o+2\pi} + E_{piston} \Big|_{\theta_o}^{\theta_o+2\pi}$$

By inspection, we can see that all the variables in $E_{linkage}$ repeat their values after one full rotation. This means that all the variables calculated at flywheel angle $\theta_o + 2\pi$ have the same value as at θ_o. Therefore,

$$E_{linkage} \Big|_{\theta_o}^{\theta_o+2\pi} = E_{linkage}(\theta_o + 2\pi) - E_{linkage}(\theta_o) = 0$$

Similarly, all the variables in E_{piston} repeat their values after one full rotation. This means that all the variables calculated at flywheel angle $\theta_o + 2\pi$ have the same value as at θ_o. Therefore,

$$E_{piston} \Big|_{\theta_o}^{\theta_o+2\pi} = E_{piston}(\theta_o + 2\pi) - E_{piston}(\theta_o) = 0$$

As a result,

$$\int_{\theta_o}^{\theta_o+2\pi} T d\theta = 0$$

Therefore, energy is conserved in the system, assuming no friction losses. If there are friction losses then

$$\int_{\theta_o}^{\theta_o+2\pi} T d\theta > 0$$

which means that net energy has to be added to the system to keep it moving. This can be shown for a specific case by introducing a friction term in equation (1), and evaluating the net energy added over one full rotation. This is left as an exercise for the reader.

There is a broader implication here. Drive mechanisms used in engines (such as piston engines), are ideally conservative, meaning that in the absence of friction no net energy is added to the system. This is true regardless of the complexity of their design. This might seem a bit counterintuitive, but if one thinks of the typical components in a drive system (such as linkages and pistons) there is an energy input in accelerating the components in one direction, but that energy is returned to the system when those same components slow down (decelerate), and then reverse direction. For example, in the crank drive shown here it takes energy to accelerate the piston to the left from the right-most position, but the piston returns energy to the system when it slows down and changes its direction of motion from left to right. This is also the reason it is necessary to keep the linkages and pistons as light as possible in a drive mechanism, because it reduces the force “peaks” taking place as these components accelerate or decelerate, as they change direction. This in turn minimizes the strain on the drive components.

The flywheel however, must be fairly heavy and have a high enough rotational inertia in order to help “smooth out” the speed fluctuations and maintain the rotation of (say, an engine) at a fairly constant speed. Problem # 8 shows an example of a flywheel calculation, illustrating this principle.

Problem # 2

An engine uses compression springs to open and close valves, using cams. Given a spring stiffness of 30,000 N/m, and a spring mass of 0.08 kg, what is the maximum engine speed to avoid "floating the valves"?

During the engine cycle the spring is compressed between 0.5 cm (valve fully closed) and 1.5 cm (valve fully open). Assume the camshaft rotates at the same speed as the engine.

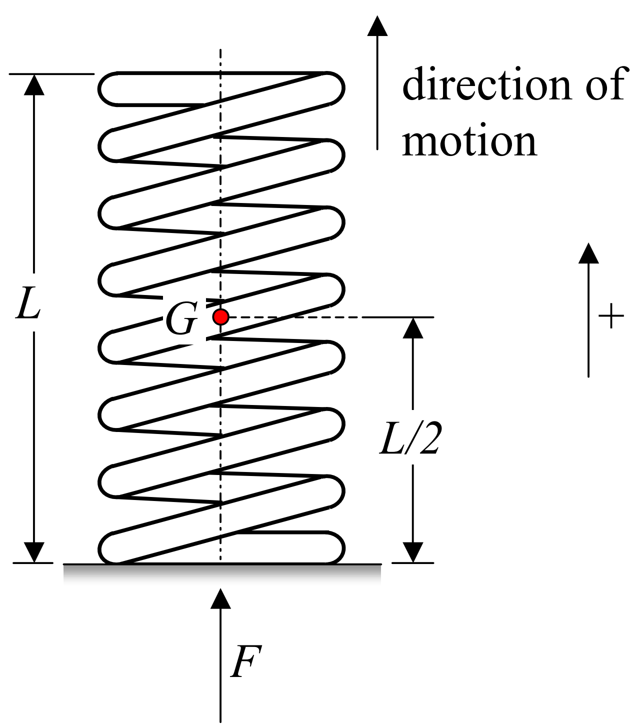
Floating the valves occurs when the engine speed is high enough so that the spring begins to lose contact with the cam when the valve closes. In other words, the spring doesn't extend quickly enough to maintain contact with the cam, when the valve closes.

For simplicity, you may assume that Hooke's Law applies to the spring, where the force acting on the spring is proportional to its amount of compression (regardless of dynamic effects).

You may ignore gravity in the calculations.

Problem # 2 Solution

We can set up a free-body diagram of the spring as shown below. For the general case we can set the spring length as *L*. Since spring geometry is almost perfectly symmetric, the center of mass of the spring (*G*) lies approximately in the middle of the spring, at a distance *L/2* as shown. Let *F* represent the force acting on the spring, due to contact with the base.



When the valve closes, the end of the spring (at distance L) must remain in contact with the cam. Once the spring begins to lose contact with the cam we have a "valve float" condition, in which the spring is unable to "keep up" with the cam, and as a result is unable to match the engine speed.

We are given that the cam speed is equal to the engine speed. Thus, the time it takes for the spring to uncompress (during valve closing) is equal to the time it takes for the engine to rotate through half a cycle. Therefore, the spring also goes through half a cycle during valve closing.

Given a spring stiffness of 30,000 N/m and a mass of 0.08 kg we wish to find the time it takes for the spring to uncompress from 1.5 cm to 0.5 cm, for the case where it is barely touching the cam.

To solve this we need to apply Newton's Second Law in the direction of motion of the spring. This law states that:

$$F = ma_G \quad (1)$$

Where:

F is the external force acting on the spring, due to contact with the base where the spring sits. This is the only force acting on the spring since the force exerted by the cam on the other end of the spring is assumed negligible during valve float (since they are barely touching).

m is the mass of the spring

a_G is the acceleration of the center of mass G of the spring

<u>Note</u>: The fact that the spring is compressing and uncompressing means that it is not a rigid body. However, the use of $F = ma_G$ does not require that a body is rigid.

Using Hooke's Law we can approximately determine the force F based on how much the spring is compressed:

$$F = k(L_o - L) \quad (2)$$

where k is the spring stiffness (a constant) and L_o is the uncompressed spring length (a constant). L is variable, where it is a function of time, so that $L = L(t)$.

By geometry, the acceleration of the center of mass G is:

$$a_G = \frac{d^2(L/2)}{dt^2} = \frac{1}{2}\frac{d^2L}{dt^2} \qquad (3)$$

Substituting (2) and (3) into (1) we get

$$k(L_o - L) = \frac{m}{2} \cdot \frac{d^2L}{dt^2}$$

This is a second order differential equation. To simplify the solution set $y = L_o - L$.

Thus,

$$ky = -\frac{m}{2} \cdot \frac{d^2y}{dt^2}$$

This can be rewritten as

$$\frac{m}{2} \cdot \frac{d^2y}{dt^2} + ky = 0$$

which becomes

$$\frac{d^2y}{dt^2} + \frac{2k}{m}y = 0$$

This equation has the known general solution:

$$y = C\sin\left(\sqrt{\frac{2k}{m}} \cdot t + \varphi\right)$$

where C and φ are constants.

Since $y = L_o - L$

$$L_o - L = C\sin\left(\sqrt{\frac{2k}{m}}\cdot t + \varphi\right)$$

To solve for C and φ we have to apply two initial conditions at time $t = 0$.

The first initial condition is: At time $t = 0$ the term $L_o - L = 1.5$ cm. Thus,

$$L_o - L = 1.5 = C\sin(\varphi)$$

The second initial condition is: At time $t = 0$ the term $dL/dt = 0$ (zero initial velocity). Thus,

$$\frac{dL}{dt} = 0 = C\sqrt{\frac{2k}{m}}\cos(\varphi)$$

Solving, we get

$$\varphi = \frac{\pi}{2}$$

and

$$C = 1.5$$

Therefore,

$$L_o - L = 1.5\sin\left(\sqrt{\frac{2k}{m}}\cdot t + \frac{\pi}{2}\right)$$

In order to determine the time it takes for the spring to uncompress from 1.5 cm to 0.5 cm we must set $L_o - L = 0.5$ in the above equation and solve for time t.

With $k = 30{,}000$ N/m and $m = 0.08$ kg we calculate $t = 0.00142$ seconds.

This is the time it takes to complete half an engine cycle.

Therefore, the time it takes to complete a full engine cycle is twice this: 0.00284 seconds. This length of time is known as the *period*.

The engine frequency f is equal to

$$f = \frac{1}{0.00284} = 352 \text{ Hz (cycles/second)}$$

Therefore, the maximum engine speed at which the valves begin to float is 352x60 = 21,120 RPM. This is a very high engine speed, and the vast majority of engines run much slower than this. Therefore, valve floating is likely not a problem.

Note: The engine speed of 21,120 RPM is actually a conservative estimate of the maximum possible engine speed. This means that the engine can run even faster before valve floating occurs. This conservative estimate of speed is a result of the assumption that the force F acting on the spring (at the base) is calculated using Hooke's Law, regardless of dynamic effects. In reality, this is not quite the case. In the presence of dynamic effects, the force F is actually somewhat larger than that given by Hooke's Law.

In order to make a more accurate model, the material properties and internal stresses of the spring would have to be accounted for, as well. This would introduce another equation, which would then be combined with Newton's Second Law equation. These two equations allow us to solve the problem more accurately. However, the accuracy of the solution would not be greatly improved.

Problem # 3

An object is traveling in a straight line. Its acceleration is given by

$$a = Ct^n$$

where C is a constant, n is a real number, and t is time.

Find the general equations for the position and velocity of the object as a function of time.

Problem # 3 Solution

This can be solved using Calculus.

Let $x(t)$ be the position of the object as a function of time.

Let $V(t)$ be the velocity of the object as a function of time.

Now,

$$a = \frac{d^2x}{dt^2} = Ct^n$$

Integrate the above equation to find the velocity of the object:

$$V(t) = \frac{dx}{dt} = \frac{C}{n+1}t^{n+1} + A$$

where A is a constant.

Integrate the above equation to find the position of the object:

$$x(t) = \frac{C}{(n+1)(n+2)}t^{n+2} + At + B$$

where B is a constant.

The constants A and B can be solved using the following initial conditions at time $t = 0$:

$x(0) = x_o$

$V(0) = V_o$

As a result, $A = V_o$ and $B = x_o$.

The position of the object as a function of time is therefore

$$x(t) = \frac{C}{(n+1)(n+2)} t^{n+2} + V_o t + x_o$$

The velocity of the object as a function of time is therefore

$$V(t) = \frac{C}{n+1} t^{n+1} + V_o$$

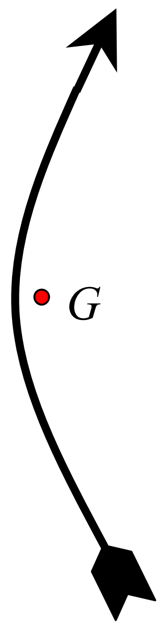
Note: If we set $n = 0$, we get $a = C$ and we obtain the familiar expressions for velocity and position given constant acceleration.

Problem # 4

In archery, when an arrow is released it can oscillate during flight. If we know the location of the center of mass of the arrow (*G*) and the shape of the arrow at an instant as it oscillates (shown below), we can determine the location of the nodes. The nodes are the "stationary" points on the arrow as it oscillates.

Using a geometric argument (no equations), determine the location of the nodes.

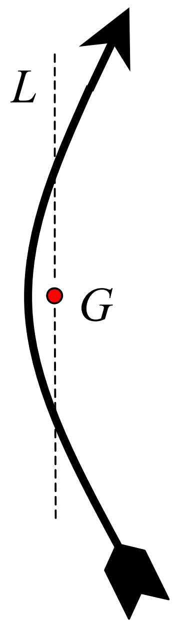
Assume that the arrow oscillates in the horizontal plane, so that no external forces act on the arrow in the plane of oscillation.



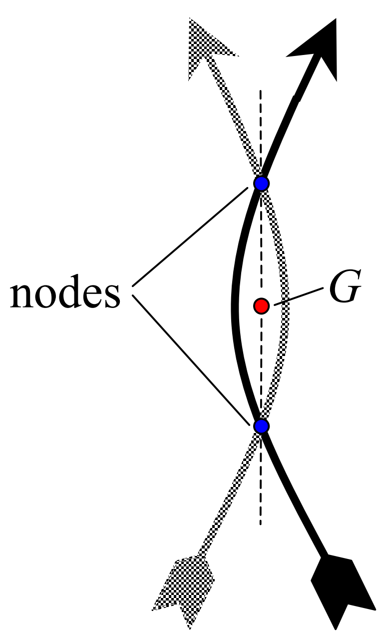
Problem # 4 Solution

The key piece of information here is that no external forces are acting on the arrow in the plane of oscillation. Therefore, for purposes of solving this problem we can treat the center of mass *G* as stationary, even though the arrow itself is oscillating. This becomes evident by Newton's Second Law, where $\sum F_{ext} = ma_G$. In the plane, $\sum F_{ext} = 0$, so $a_G = 0$. This means that the center of mass *G* of the arrow is either moving at a constant horizontal velocity in a straight line, or *G* is stationary. Although the former is true, for simplicity purposes we can treat the center of mass *G* as stationary since it will help us visualize what is happening.

The first step is to draw a line L passing through G and aligned with the direction of motion of the arrow. This line is also the symmetry line of the arrow as it oscillates.



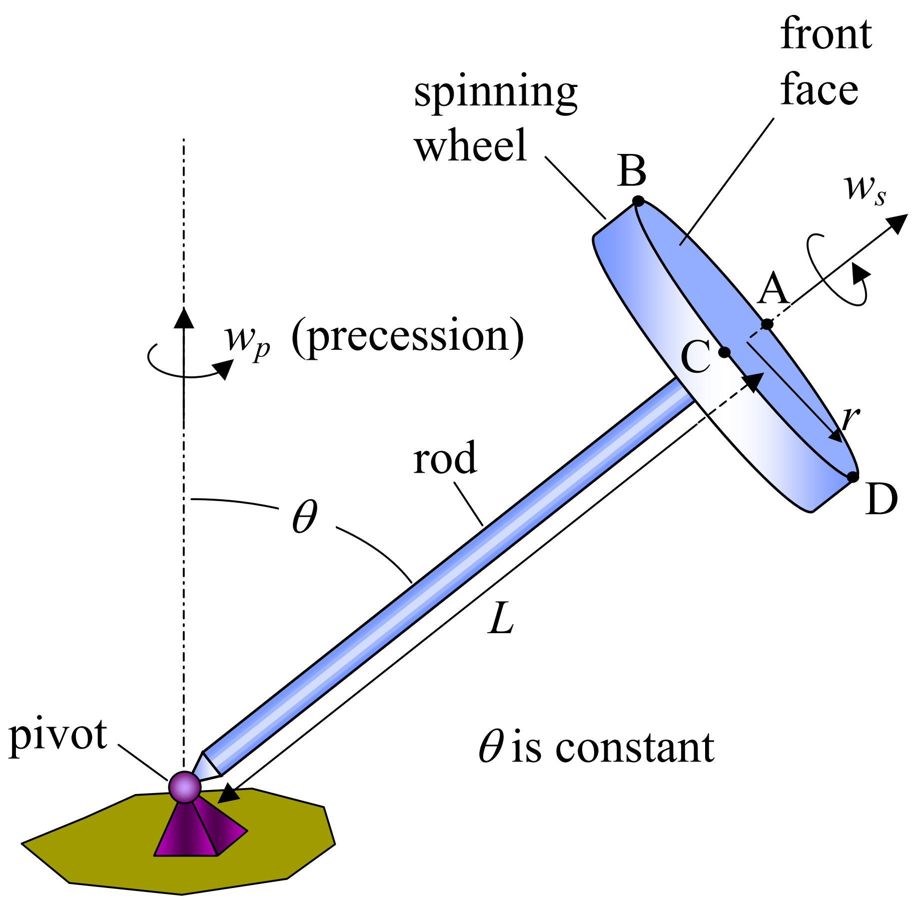
The second step is to reflect the arrow about this symmetry line. The nodes are the points of intersection of the arrow (shown above) with the reflected arrow. We are done.



Problem # 5

A gyroscope wheel is spinning at a constant angular velocity w_s while precessing about a vertical axis at a constant angular velocity w_p. The distance from the pivot to the center of the front face of the spinning gyroscope wheel is L, and the radius of the wheel is r. The rod connecting the pivot to the wheel makes a constant angle θ with the vertical.

Determine the acceleration components normal to the wheel, at points A, B, C, D labeled as shown.



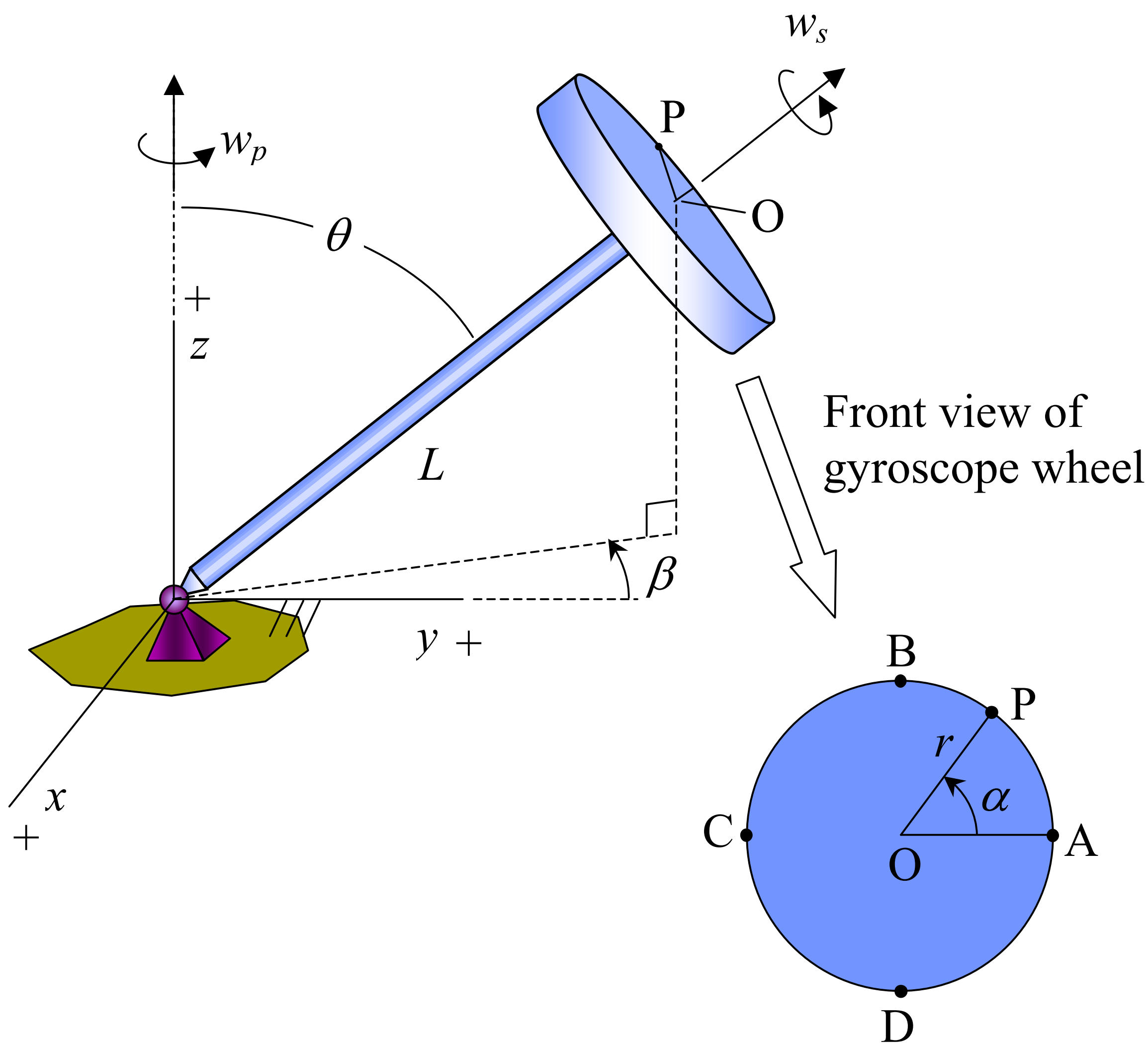
Problem # 5 Solution

To solve this problem it is useful to introduce a reference frame *xyz* (fixed to ground) with origin at the pivot, and define two new variables α and β, as shown in the schematic below.

Point O is defined as the center of the wheel, on the front face.

The angle α is in the plane of the gyroscope wheel, and the angle β is in the *xy* plane.

By definition, $w_s = d\alpha/dt$ and $w_p = d\beta/dt$.



Define x_p, y_p, z_p as the coordinates of point P on the front face of the gyroscope wheel, relative to the reference frame xyz.

By close inspection,

$$x_p = -L\sin\theta\sin\beta - r\cos\alpha\cos\beta + r\cos\theta\sin\alpha\sin\beta$$
$$y_p = L\sin\theta\cos\beta - r\cos\alpha\sin\beta - r\cos\theta\sin\alpha\cos\beta$$
$$z_p = L\cos\theta + r\sin\alpha\sin\theta$$

To determine the acceleration components normal to the gyroscope wheel, without loss of generality, we can evaluate the second derivative of y_p and z_p (with respect to time) at $\beta = 0$, and resolve them in the direction normal to the wheel, at each of the locations A, B, C, D. Note that we do not need to determine the second derivative of x_p when $\beta = 0$. This is because the normal to the wheel has no x-component when $\beta = 0$. This saves us a bit of work.

In mathematical terms, we wish to determine the normal acceleration component a_n at the locations A, B, C, D where

$$a_n = \sin\theta\frac{d^2y_p}{dt^2} + \cos\theta\frac{d^2z_p}{dt^2}$$

For point A, $\alpha = 0°$

For point B, $\alpha = 90°$

For point C, $\alpha = 180°$

For point D, $\alpha = 270°$

To simplify the notation set

$$\frac{d^2 y_p}{dt^2} \equiv \ddot{y}_p$$

$$\frac{d^2 z_p}{dt^2} \equiv \ddot{z}_p$$

$$\frac{d\alpha}{dt} \equiv \dot{\alpha}$$

$$\frac{d^2\alpha}{dt^2} \equiv \ddot{\alpha}$$

$$\frac{d\beta}{dt} \equiv \dot{\beta}$$

$$\frac{d^2\beta}{dt^2} \equiv \ddot{\beta}$$

Now, from the problem statement the angular velocities can be expressed as:

$$\dot{\alpha} = w_s$$

$$\dot{\beta} = w_p$$

and since the angular velocities are constant

$$\ddot{\alpha} = 0$$

$$\ddot{\beta} = 0$$

We are now ready to evaluate the second derivative of y_p and z_p.

$$\ddot{y}_p = -L\cos\beta\sin\theta\cdot w_p^2 + rw_p(\sin\alpha\cos\beta\cdot w_s + \cos\alpha\sin\beta\cdot w_p)$$
$$+ rw_s(\cos\alpha\sin\beta\cdot w_s + \sin\alpha\cos\beta\cdot w_p)$$
$$+ r\cos\theta\cdot w_s(\sin\alpha\cos\beta\cdot w_s + \cos\alpha\sin\beta\cdot w_p)$$
$$+ r\cos\theta\cdot w_p(\cos\alpha\sin\beta\cdot w_s + \sin\alpha\cos\beta\cdot w_p)$$

$$\ddot{z}_p = -r\sin\alpha\sin\theta\cdot w_s^2$$

At $\beta = 0$ we have

$$\ddot{y}_p = -L\sin\theta\cdot w_p^2 + 2rw_p w_s\sin\alpha + r\cos\theta\cdot w_s^2\sin\alpha$$
$$+ r\cos\theta\cdot w_p^2\sin\alpha$$

$$\ddot{z}_p = -r\sin\alpha\sin\theta\cdot w_s^2$$

For point A, $\alpha = 0°$ and we have

$$\ddot{y}_p = -L\sin\theta\cdot w_p^2$$

$$\ddot{z}_p = 0$$

Since

$$a_n = \sin\theta\cdot\ddot{y}_p + \cos\theta\cdot\ddot{z}_p$$

The normal acceleration component at A is:

$$a_n = -L\sin^2\theta \cdot w_p^2$$

For point B, $\alpha = 90°$ and we have

$$\ddot{y}_p = -L\sin\theta \cdot w_p^2 + 2rw_p w_s + r\cos\theta \cdot w_s^2 + r\cos\theta \cdot w_p^2$$

$$\ddot{z}_p = -r\sin\theta \cdot w_s^2$$

Since

$$a_n = \sin\theta \cdot \ddot{y}_p + \cos\theta \cdot \ddot{z}_p$$

The normal acceleration component at B is:

$$a_n = -L\sin^2\theta \cdot w_p^2 + 2rw_p w_s \sin\theta + r\sin\theta\cos\theta \cdot w_p^2$$

For point C, $\alpha = 180°$ and we have

$$\ddot{y}_p = -L\sin\theta \cdot w_p^2$$

$$\ddot{z}_p = 0$$

Since

$$a_n = \sin\theta \cdot \ddot{y}_p + \cos\theta \cdot \ddot{z}_p$$

The normal acceleration component at C is:

$$a_n = -L\sin^2\theta \cdot w_p^2$$

For point D, $\alpha = 270°$ and we have

$$\ddot{y}_p = -L\sin\theta \cdot w_p^2 - 2rw_p w_s - r\cos\theta \cdot w_s^2 - r\cos\theta \cdot w_p^2$$

$$\ddot{z}_p = r\sin\theta \cdot w_s^2$$

Since

$$a_n = \sin\theta \cdot \ddot{y}_p + \cos\theta \cdot \ddot{z}_p$$

The normal acceleration component at D is:

$$a_n = -L\sin^2\theta \cdot w_p^2 - 2rw_p w_s \sin\theta - r\sin\theta\cos\theta \cdot w_p^2$$

If the normal acceleration components are positive that means they point outwards, away from the pivot. If the acceleration components are negative that means they point inwards, towards the pivot.

The above results indicate that the normal components of acceleration vary along the gyroscope wheel. This acceleration "gradient" helps to explain why a spinning gyroscope precesses and does not fall down due to gravity. This is explained in more detail here: http://www.real-world-physics-problems.com/gyroscope-physics.html.

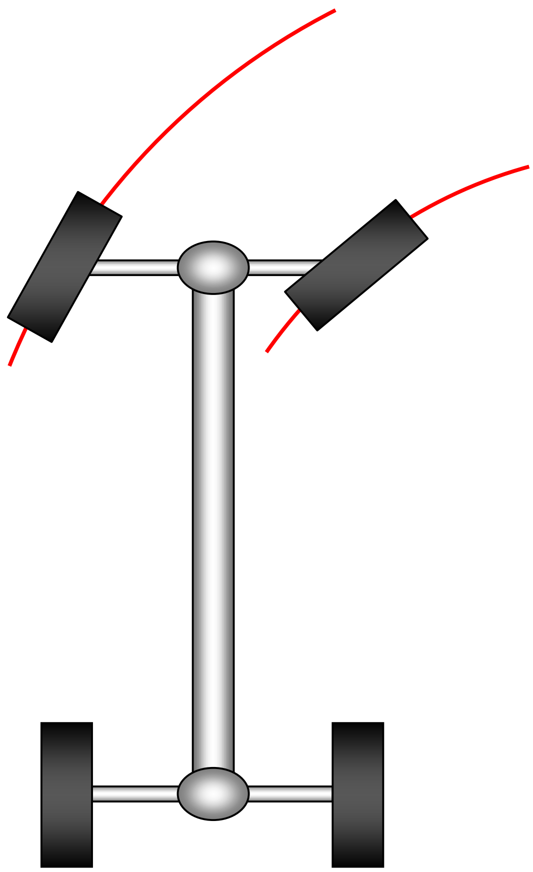
The normal acceleration components can also be calculated (perhaps more easily) by using vector equations for general motion, given here: http://www.real-world-physics-problems.com/general-motion.html.

However, by finding equations for the position (x_p, y_p, z_p) with respect to ground, and then taking the second derivative of the position with respect to time, we make the acceleration calculation somewhat "automatic"; which is a convenience, even though it does get algebraically "messy".

Problem # 6

When a vehicle makes a turn, the two front wheels trace out two arcs as shown in the figure below. The wheel facing towards the inside of the turn has a steering angle that is greater than that of the outer wheel. This is necessary to ensure that both front wheels smoothly trace out two arcs, which have the same center, otherwise the front wheels will skid on the ground during the turn.

During a turn, do the rear wheels necessarily trace out the same arcs as the front wheels? Based on your answer, what are the implications for making a turn close to the curb?

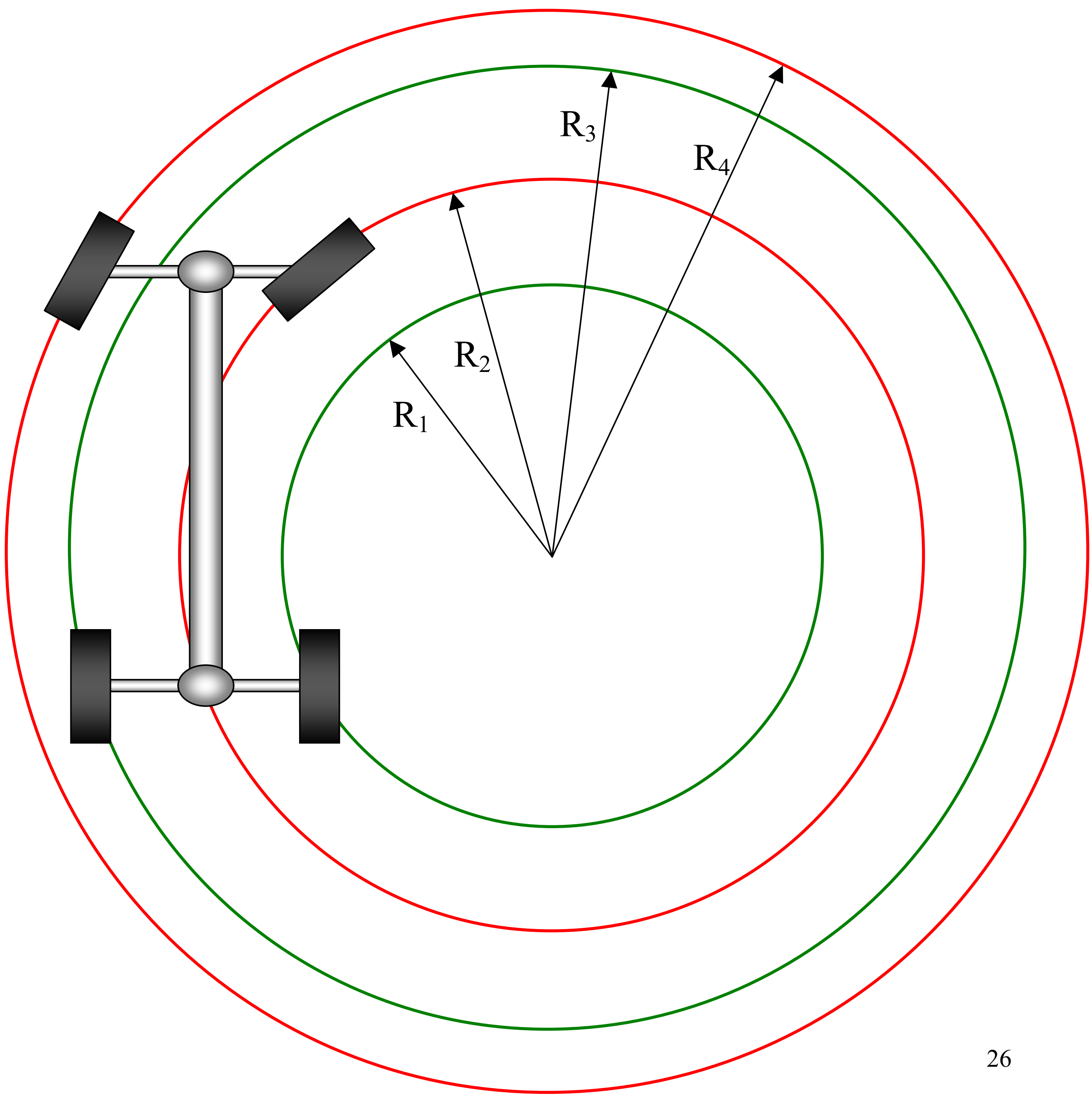


Problem # 6 Solution

To help visualize the problem, draw the circles that would be traced out by the four wheels if the vehicle were to go around in a continuous loop.

As you can see in this example, the circles traced by the rear wheels lie on the inside of the circles traced by the front wheels, so in general they do not follow the same arcs.

This illustration shows that if you turn close to the curb, the rear wheel on the inside of the turn may hit the curb, even if the front wheel misses it. But whether this happens or not depends on the length of the vehicle and the radius of the turn.



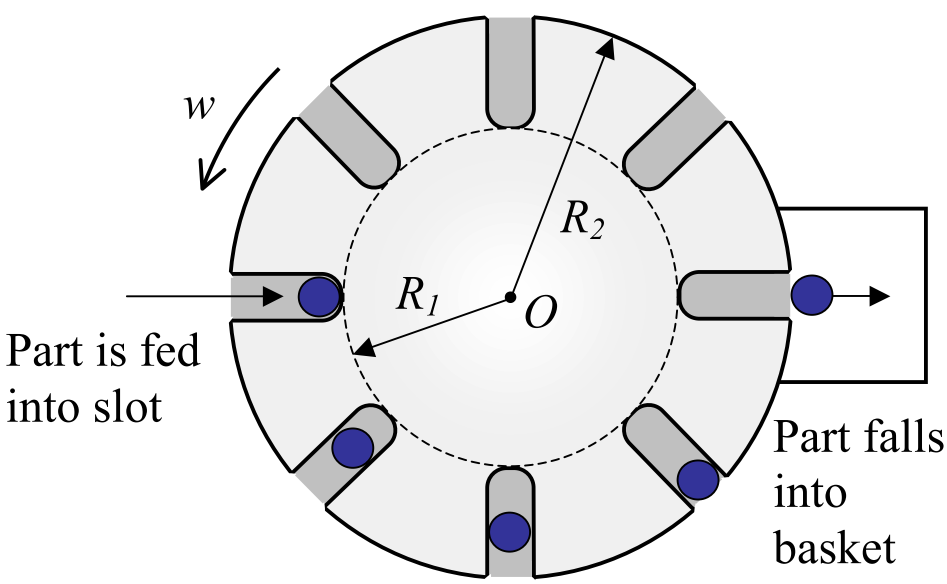
Problem # 7

A horizontal turntable at an industrial plant is continuously fed parts into a slot (shown on the left). It then drops these parts into a basket (shown on the right). The turntable rotates 180° between these two stages. The turntable briefly stops at each 1/8th of a turn in order to receive a new part into the slot on the left.

If the rotational speed of the turntable is *w* radians/second, and the outer radius of the turntable is R_2, what must be the inner radius R_1 so that the parts fall out of the slot and into the basket, as shown?

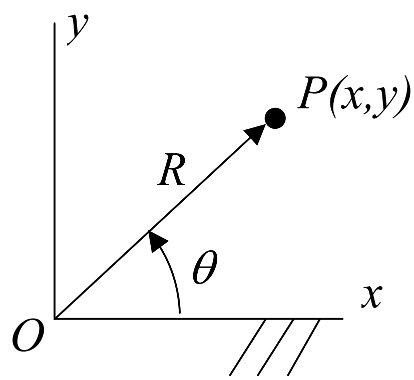
Assume:

- The angular speed *w* of the turntable can be treated as constant and continuous; which means you can ignore the brief stops the turntable makes at each 1/8th of a turn.
- The location of the basket is 180° from the feed location.
- The slots are very well lubricated so that there is no friction between the slot and part.
- The parts can be treated as particles, which means you can ignore their dimensions in the calculation.
- The slots are aligned with the radial direction of the turntable.



Problem # 7 Solution

It is useful to first derive a general expression for the radial acceleration of a particle *P* using a polar coordinate system as defined in the figure below.



We wish to find a general equation for the acceleration of particle *P* along the radial *R*-direction. This equation can then be applied to the solution of this problem, which involves motion of the parts along the slots, which are aligned with the radial direction of the turntable.

Without loss of generality, one of the easiest ways to find this general equation is to determine the acceleration of particle *P* along the *x*-direction when $\theta = 0$. This will be equal to the radial acceleration.

First, set

$$x(t) = R\cos\theta$$

Next, evaluate the second derivative of $x(t)$ with respect to time, where $d\theta/dt = w$ and $d^2\theta/dt^2 = 0$ (since w is constant). The second derivative of $x(t)$ at $\theta = 0$ is the radial acceleration.

We have

$$\frac{d^2x(t)}{dt^2} = \frac{d^2R}{dt^2}\cos\theta - R\cos\theta\cdot w^2 - 2\frac{dR}{dt}\sin\theta\cdot w$$

At $\theta = 0$

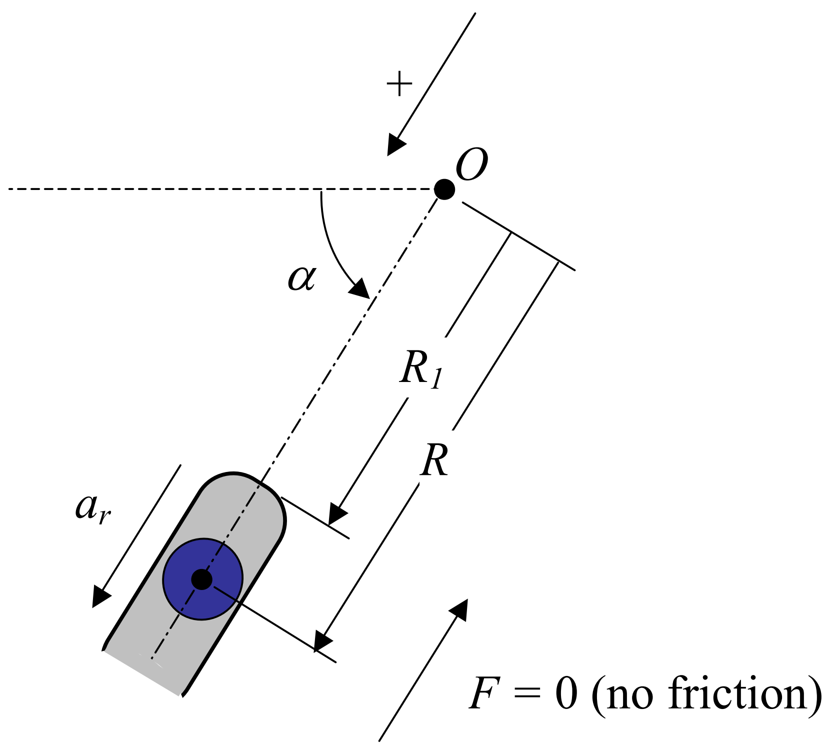
$$\frac{d^2x(t)}{dt^2} = \frac{d^2R}{dt^2} - Rw^2$$

This is the general equation for radial acceleration of particle *P*.

The next step is to apply Newton's Second Law in the radial direction:

$$F = ma_r$$

where *F* is the force acting on the particle in the radial direction, *m* is the mass of the particle, and a_r is the radial acceleration of the particle. The free body diagram below illustrates this. Note that α (shown below) is defined as the angle of rotation of the turntable, where $\alpha = 0$ is the location where the part is fed into the slot, and $\alpha = 180°$ is the location where the part drops into the basket.



Now,

$$a_r = \frac{d^2 R}{dt^2} - Rw^2$$

Substitute this into the equation for Newton's Second Law and we get

$$F = m\left(\frac{d^2 R}{dt^2} - Rw^2\right)$$

Since there is no friction, $F = 0$. Therefore

$$a_r = 0 = \frac{d^2 R}{dt^2} - Rw^2$$

This is a second order differential equation in terms of R. It has the general solution

$$R = C_1 e^{wt} + C_2 e^{-wt}$$

where t is time, and C_1 and C_2 are constants that can be solved based on initial conditions at time $t = 0$ (corresponding to $\alpha = 0$).

At time $t = 0$, $R = R_1$. Thus,

$$R_1 = C_1 + C_2 \qquad (1)$$

At time $t = 0$, $dR/dt = 0$ (the part starts moving from rest). This gives us

$$\frac{dR}{dt} = C_1 w e^{wt} - C_2 w e^{-wt}$$

Thus, at time $t = 0$

$$0 = C_1 - C_2 \qquad (2)$$

Solving (1) and (2) we get

$$C_1 = C_2 = \frac{R_1}{2}$$

Therefore,

$$R = \frac{R_1}{2} e^{wt} + \frac{R_1}{2} e^{-wt}$$

For α = 180° (half a rotation) we want $R = R_2$ so that the part falls off the turntable and into the basket. The time t it takes for the turntable to rotate α = 180° is equal to $t = \pi/w$ (where π is equal to 180°, in radians).

Substituting $t = \pi/w$ into the above equation and equating it to R_2 we have

$$R_2 = \frac{R_1}{2} e^{\pi} + \frac{R_1}{2} e^{-\pi}$$

Solving for R_1 we get

$$R_1 = \frac{2R_2}{e^{\pi} + e^{-\pi}}$$

This is the necessary inner radius so that the parts fall out of the slots when the slots reach the location of the basket (at α = 180°).

Now, it might seem a bit odd that even though there is no force acting on the parts (no friction), the parts still move outwards as the turntable spins. The reason for this is due to the kinematics of the problem. The parts cannot experience radial acceleration since they have no radial force acting on them, and the only way for this to be true is if the parts move outward in the radial direction (while constrained by the sides of the slots). So the parts appear to be pushed outwards due to an imaginary force. This imaginary force is sometimes called a fictitious *centrifugal force*.

Also, it is interesting that R_1 does not depend on the rotation speed w. What is the reason for this?

Problem # 8

A flywheel for a single piston engine rotates at an average speed of 1500 RPM. During half a rotation the flywheel has to absorb 1000 J of energy. If the maximum permissible speed fluctuation is ± 60 RPM, what is the minimum rotational inertia of the flywheel? Assume there is no friction.

Problem # 8 Solution

For a rotating flywheel the kinetic energy *KE* is given by

$$KE = \frac{1}{2} I_o w^2$$

where I_o is the rotational inertia of the flywheel rotating about a fixed point *O* (the center of the flywheel), and *w* is the angular velocity of the flywheel, in radians/second.

Since the maximum permissible speed fluctuation is ± 60 RPM, we can say that the maximum engine speed is 1560 RPM, and the minimum engine speed is 1440 RPM (this gives an average speed of 1500 RPM).

Now, 1560 RPM = 163.36 radians/second, and 1440 RPM = 150.8 radians/second. The change in kinetic energy of the flywheel is the difference in kinetic energy between these two speeds. If we assume no friction, the energy absorbed by the flywheel is equal to its change in kinetic energy.

We can set up the following equality, and solve for I_o.

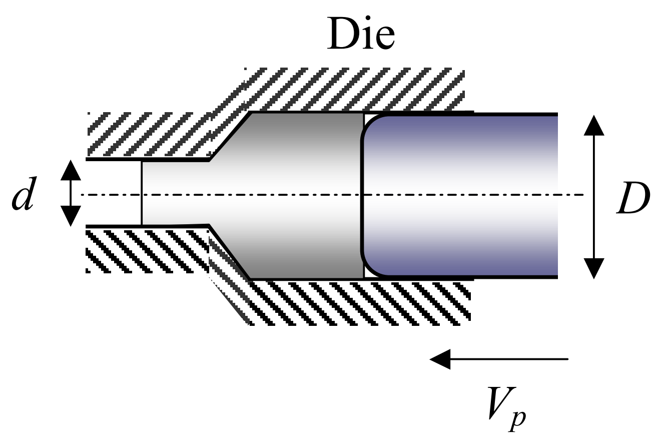
$$1000 = \frac{1}{2} I_o (163.36)^2 - \frac{1}{2} I_o (150.8)^2$$

Solving, we get $I_o = 0.51\ \text{kg}\cdot\text{m}^2$.

This is the minimum rotational inertia of the flywheel. A higher rotational inertia will result in a speed fluctuation less than ± 60 RPM.

Problem # 9

An aluminum extrusion process is simulated numerically with a computer. In this process, a punch pushes an aluminum billet of diameter D through a die of smaller diameter d. In the computer simulation, what is the maximum punch velocity V_p so that the net dynamic force (predicted by the simulation) acting on the aluminum during extrusion is at most 5% of the force due to deformation of the aluminum? Evaluate for a specific case where $D = 0.10$ m, $d = 0.02$ m, and the density of aluminum is $\rho = 2700$ kg/m^3.



The force due to deformation of the aluminum during extrusion is given by

$$F_{def} = \frac{\begin{pmatrix} \pi D^2 \times 140 \times 10^6 \times \left(\ln(D^2/d^2)\right)^{0.25} \\ \times \left(0.8 + 1.2\ln(D^2/d^2)\right) \end{pmatrix}}{5}$$

Hint:

The extrusion of the aluminum through the die is analogous to fluid flowing through a pipe which transitions from a larger diameter to a smaller diameter (e.g. water flowing through a fireman's hose). The net dynamic force acting on the fluid is the net force required to accelerate the fluid, which occurs when the velocity of the fluid increases as it flows from the larger diameter section to the smaller diameter section (due to conservation of mass).

Problem # 9 Solution

Since the aluminum "flows" through the die, the physics is essentially the same as that of a fluid flowing through a pipe that decreases in diameter.

The net dynamic force acting on the aluminum during extrusion is given by

$$F_{net} = \frac{dm}{dt}\left(V_2 - V_1\right) \quad (1)$$

where *dm/dt* is the mass flow rate, V_1 is the initial velocity of the aluminum billet before it passes through the die, and V_2 is the final velocity of the billet after it has passed through the die. Note that $V_1 = V_p$.

The mass flow rate is given by

$$\frac{dm}{dt} = \rho V_p A_1$$

where A_1 is the initial area, and

$$A_1 = \frac{\pi}{4} D^2$$

Therefore,

$$\frac{dm}{dt} = \rho V_p \frac{\pi}{4} D^2$$

V_2 can be determined by mass conservation, the same way as for a fluid flow problem. This means that

$$V_p A_1 = V_2 A_2 \quad (2)$$

where

$$A_2 = \frac{\pi}{4} d^2$$

Substituting A_1 and A_2 into equation (2) we have

$$V_p \frac{\pi}{4} D^2 = V_2 \frac{\pi}{4} d^2$$

Thus,

$$V_2 = V_p \frac{D^2}{d^2}$$

Therefore, from equation (1), the net dynamic force acting on the aluminum during extrusion is

$$F_{net} = \rho V_p^2 \frac{\pi}{4} D^2 \left(\frac{D^2}{d^2} - 1 \right)$$

As given in the problem statement, the maximum V_p is such that $F_{net} = 0.05 F_{def}$.

It follows from the above equation that the maximum punch velocity in the simulation is given by

$$V_{p,\max} = \sqrt{\frac{0.05 F_{def}}{\rho \frac{\pi}{4} D^2 \left(\frac{D^2}{d^2} - 1 \right)}}$$

Evaluate for a specific case where $D = 0.10$ m, $d = 0.02$ m, and $\rho = 2700$ kg/m^3.

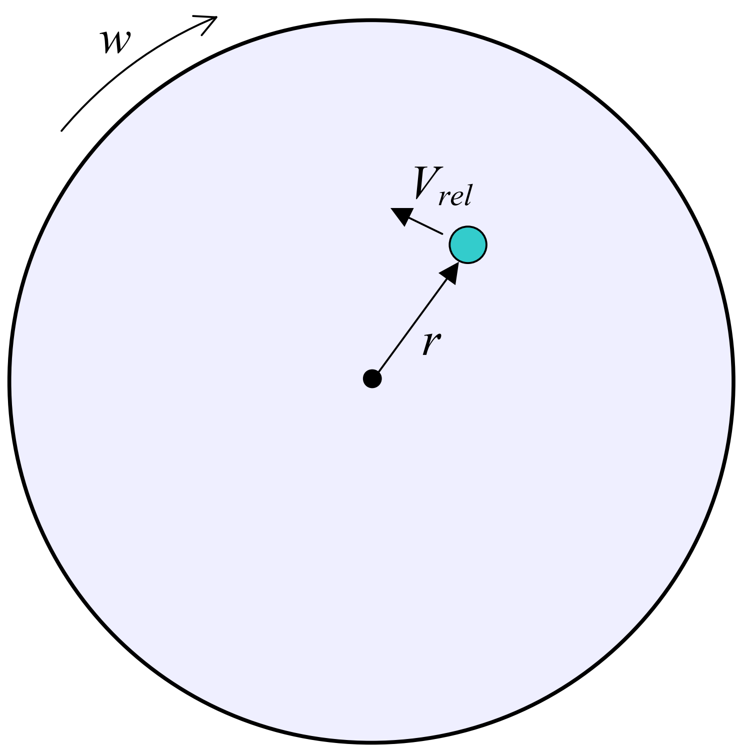
We get $V_{p,max} = 23.2$ m/s.

When simulating a process on a computer it is necessary to keep the computation time as short as possible. In this case, where we are simulating an aluminum extrusion process, a short computation time can only be achieved by using a high punch speed V_p in the simulation. But unfortunately, this can result in artificially high over-predictions for the forces acting on the punch and die, since dynamic effects become more significant for greater punch velocity. So it is necessary to limit the punch velocity to avoid excessive dynamic effects in the force predictions.

Problem # 10

A child on a horizontal merry-go-round (shown below) gives an initial velocity V_{rel} to a ball. Find the initial direction and velocity V_{rel} of the ball relative to the merry-go-round so that, relative to the child, the ball goes around in a perfect circle as he's sitting on the merry-go-round. Assume there is no friction between merry-go-round and ball.

The merry-go-round is rotating at a constant angular velocity of *w* radians/second, and the ball is released at a radius *r* from the center of the merry-go-round.



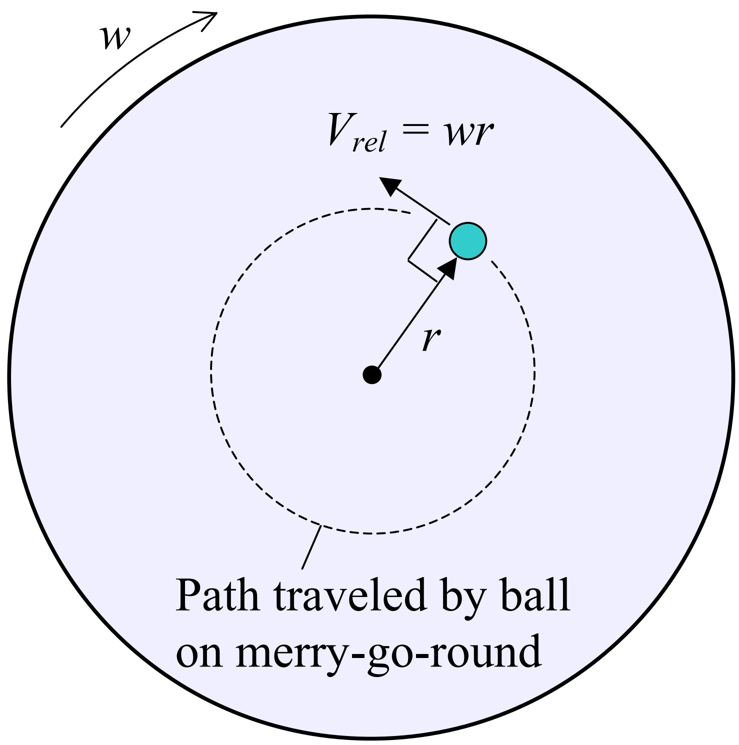
Problem # 10 Solution

Since there is no friction between the ball and merry-go-round, the general motion of the ball is such that it will either be stationary or travel in a straight line relative to fixed ground (in other words, as observed by someone standing on the ground). This happens because there are no net external forces acting on the ball (Newton's First Law).

However, relative to the child sitting on the merry-go-round, the motion of the ball follows a curved path. We want to find the initial velocity and direction of the ball such that the path traced on the merry-go-round (by the ball) is a perfect circle.

A perfect circle means that the ball returns to its starting point after one full rotation of the merry-go-round. This means that the ball must have a velocity of zero relative to fixed ground. The ball will appear to be a stationary dot (relative to ground) coinciding with the initial location of the child. As the merry-go-round turns, the child sees this "dot" moving away from him. And after one full revolution he is once more coincident with the "dot".

Therefore, in order to have zero velocity of the ball relative to fixed ground, the child must roll the ball away from him at an angle of 90°, opposite the direction of rotation of the merry-go-round, and with a relative speed equal to $V_{rel} = wr$ (see figure below). This will cancel out the tangential velocity of the merry-go-round at the location of the ball (equal to wr), and the resulting velocity of the ball will be zero relative to fixed ground.



Problem # 11

A heavy pump casing with a mass *m* is to be lifted off the ground using a crane. For simplicity, the motion is assumed to be two-dimensional, and the pump casing is represented by a rectangle having side dimensions *ab* (see figure). A cable of length L_1 is attached to the crane (at point *P*) and the pump casing (at point *O*). The crane pulls up vertically on the cable with a constant velocity V_p.

The center of mass *G* of the pump casing is assumed to lie in the center of the rectangle. It is located at a distance L_2 from point *O*. The right side of the pump casing is located at a horizontal distance *c* from the vertical line passing though point *P*.

Find the maximum cable tension during the lift, which includes the part of the lift before the pump casing loses contact with the ground, and after the pump casing loses contact with the ground (lift off). In this stage the pump casing swings back and forth.

Evaluate for a specific case where:

$a = 0.4$ m

$b = 0.6$ m

$c = 0.2$ m

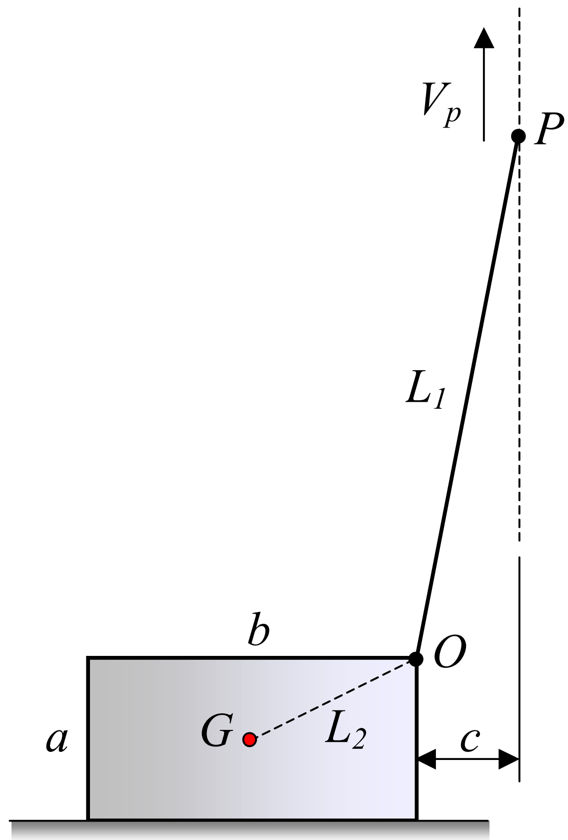
$L_1 = 3$ m

$m = 200$ kg

$I_G = 9$ kg·m^2 (rotational inertia of pump casing about *G*)

Assume:

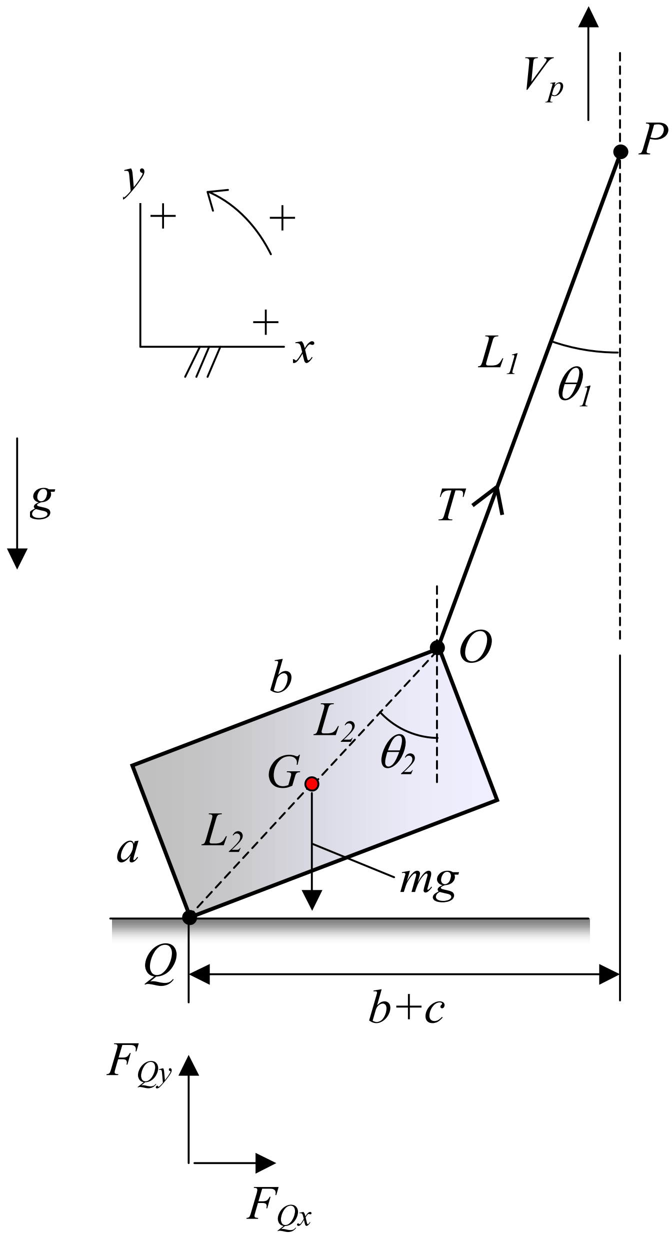
- The friction between the pump casing and ground is high enough so that the pump casing does not slide along the ground (towards the right), before lift off occurs.
- Before lift off occurs, dynamic effects are negligible.
- The velocity V_p is fast enough so that the bottom of the pump casing swings clear of the ground after lift off occurs.
- For purposes of approximating the cable tension, you can model the system as a regular pendulum during swinging (you can ignore double pendulum effects).
- The mass of the cable can be neglected.



Problem # 11 Solution

There are two distinct cases to consider. The first case is when the pump casing is being raised off the ground. This case can be examined using a schematic shown in the figure below. The following new variables are introduced:

- T is the cable tension
- Q represents the pivot point for the pump casing during the lift
- mg is the weight of the casing which acts through the center of mass G
- θ_1, θ_2 are angles given as shown



The pump casing lifts off when the vertical component of the cable tension T exceeds the weight mg. At this instant, $F_{Qy} = 0$.

To determine when lift off occurs we can take the sum of the moments about point Q. This allows us to determine the tension T at lift off.

Since this part of the lift is assumed to be approximately static (negligible dynamic effects), the sum of the moments is zero. Thus,

$$0 = -mgL_2 \sin\theta_2 + T(2L_2)\sin(\theta_2 - \theta_1) \quad (1)$$

By geometry,

$$\frac{b + c - 2L_2 \sin\theta_2}{L_1} = \sin\theta_1 \quad (2)$$

By the Pythagorean theorem,

$$(2L_2)^2 = a^2 + b^2 \quad (3)$$

When lift off occurs $F_{Qy} = 0$, which means that

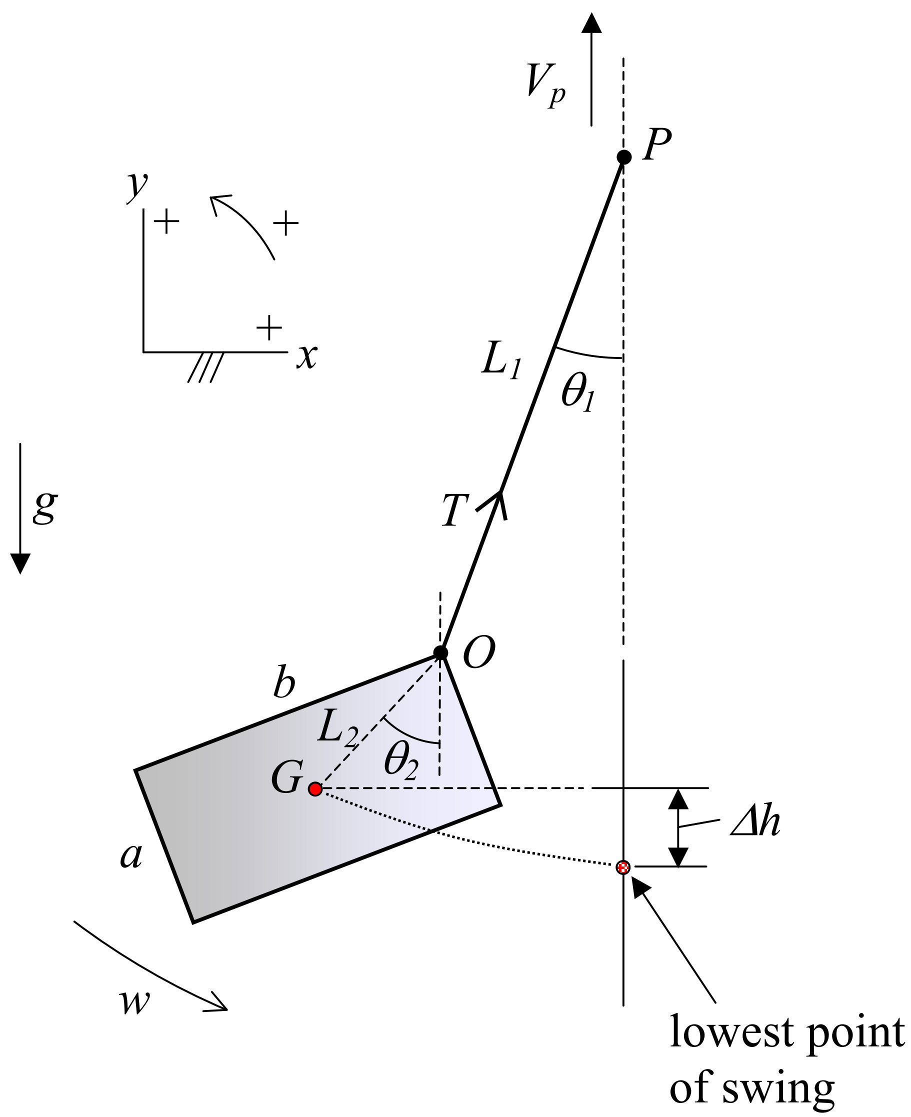
$$T\cos\theta_1 = mg \quad (4)$$

With $g = 9.8$ m/s^2 solve equations (1), (2), (3), (4). We get: $T = 1994$ N, $\theta_1 = 10.5°$, $\theta_2 = 20.4°$. These are the values at lift off.

For the second case we assume that for purposes of approximating the maximum cable tension, the system can be approximated as a regular pendulum. The figure below illustrates the starting position of the swing (just after lift off occurs).

As an approximation we can say that $\theta_1 \cong \theta_2 = 20.4°$ (the initial start position from rest, at the start of the swing). As a result, the system will behave as a regular pendulum, and double pendulum effects can be neglected. This will allow us to approximate the cable tension with minimal mathematical effort.

For this second case, the maximum cable tension will occur at the lowest point in the swing.



In the above figure, the variable w represents the angular velocity of the pump casing, and Δh represents the change in vertical height of the center of mass G of the casing, measured from the lowest point in the swing (the datum). At height Δh the angular velocity of the casing is zero since this is the highest point of the swing (and also where the swing starts).

The easiest way to find the tension at the lowest point of the swing is to use an energy method, where the change in gravitational potential energy of the pump casing is equal to the gain in kinetic energy of the pump casing. Now, since the velocity V_p is constant, the acceleration of P is zero and there are no dynamic effects due to the motion of P. This means that for calculation purposes the system can be treated as if P were a fixed point (i.e. $V_p = 0$).

As a result, we can equate the change in gravitational potential energy due to the change in height of the mass center of the casing (Δh), to the kinetic energy of the casing at the lowest point in the swing (with angular velocity w). Thus,

$$mg\Delta h = \frac{1}{2} I_p w^2 \qquad (5)$$

where I_p is the rotational inertia of the pump casing about an axis pointing out of the page and passing through point P.

From the parallel axis theorem,

$$I_p = I_G + m(L_1 + L_2)^2$$

From geometry,

$$\Delta h = L_1 + L_2 - (L_1 + L_2)\cos\theta_2$$

where $\theta_2 \cong \theta_1 = 20.4°$ (from before).

Solving for w in equation (5) we get $w = 0.60$ radians/second.

To find the tension at the lowest point in the swing apply Newton's Second Law:

$$F = ma_G$$

where F is the sum of the forces acting on the pump casing at the lowest point, and a_G is the acceleration of the center of mass G of the casing. Now,

$$F = -mg + T$$

a_G is equal to the centripetal acceleration, which points upward, and is given by

$$a_G = w^2 R$$

where R is the radius of the arc traced by point G, where

$$R = L_1 + L_2$$

Substituting the above three equations into the equation for Newton's Second Law and solving for T we get

$$T = mw^2 (L_1 + L_2) + mg$$

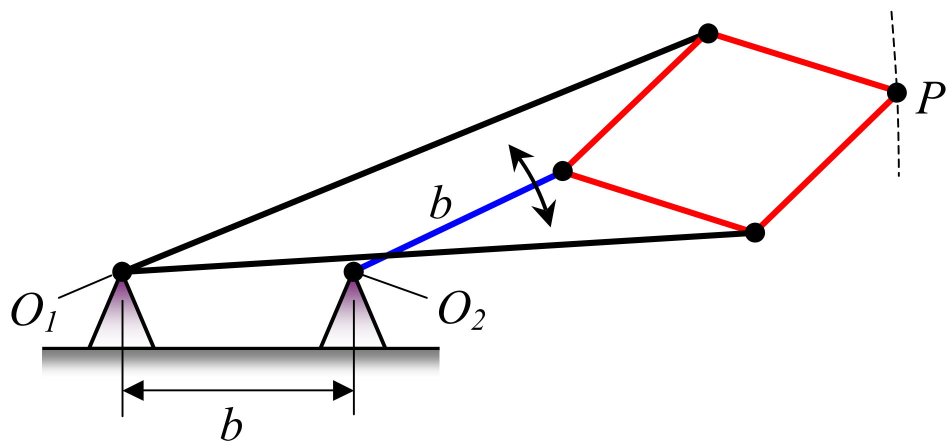
Solving, we get $T = 2205$ N. This is the maximum cable tension during the lift.

It is worth mentioning that if we were to solve this problem more exactly, by treating the system as a double pendulum with $\theta_1 = 10.5°$ and $\theta_2 = 20.4°$, the maximum cable tension would be equal to 2204 N. This is almost exactly the same!

Problem # 12

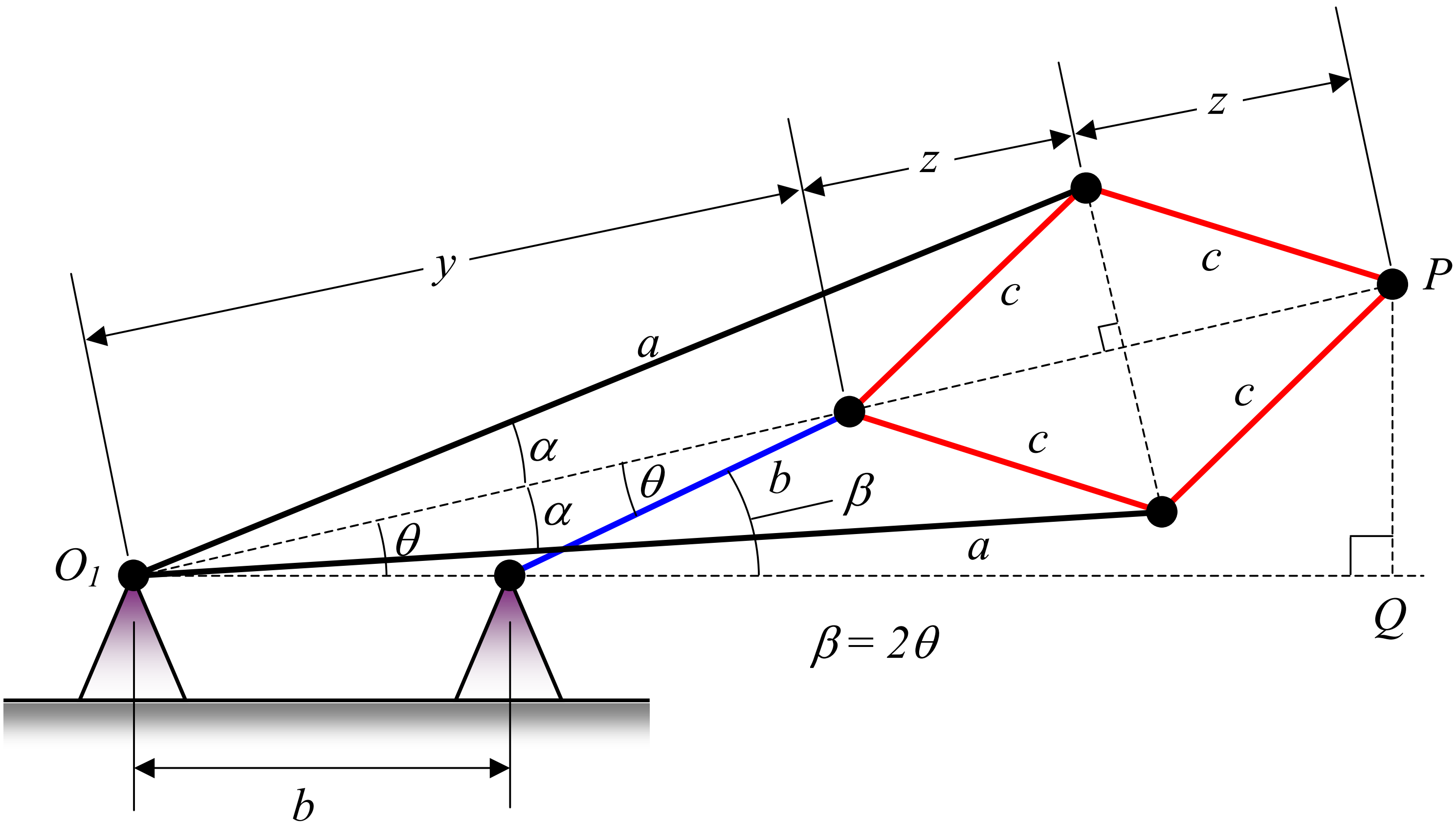
A linkage arrangement is shown below. The pin joints O_1 and O_2 are attached to a stationary base and are separated by a distance b. The linkages of identical color have the same length. All linkages are pin jointed and allow for rotation. Determine the path traced by the end point P as the blue linkage of length b rotates back and forth.

Why is this result interesting?



Problem # 12 Solution

To solve this problem we need to analyze the geometry, with new variables introduced as shown. Symmetry plays a large part in the analysis.



By the law of cosines,

$$c^2 = a^2 + y^2 - 2ay\cos\alpha \qquad (1)$$

where

$$y = 2b\cos\theta \qquad (2)$$

In addition,

$$z = a\cos\alpha - y \qquad (3)$$

To determine the path traced by P find the distance O_1Q as a function of angle θ.

From geometry,

$$O_1Q = (y + 2z)\cos\theta \qquad (4)$$

Substituting equations (1), (2), (3) into equation (4) we get

$$O_1Q = \frac{a^2 - c^2}{2b}$$

This result is very interesting because it shows that O_1Q is a constant, which is independent of θ. This result tells us that, as the blue linkage of length b rotates back and forth, the path traced by P is a straight line!

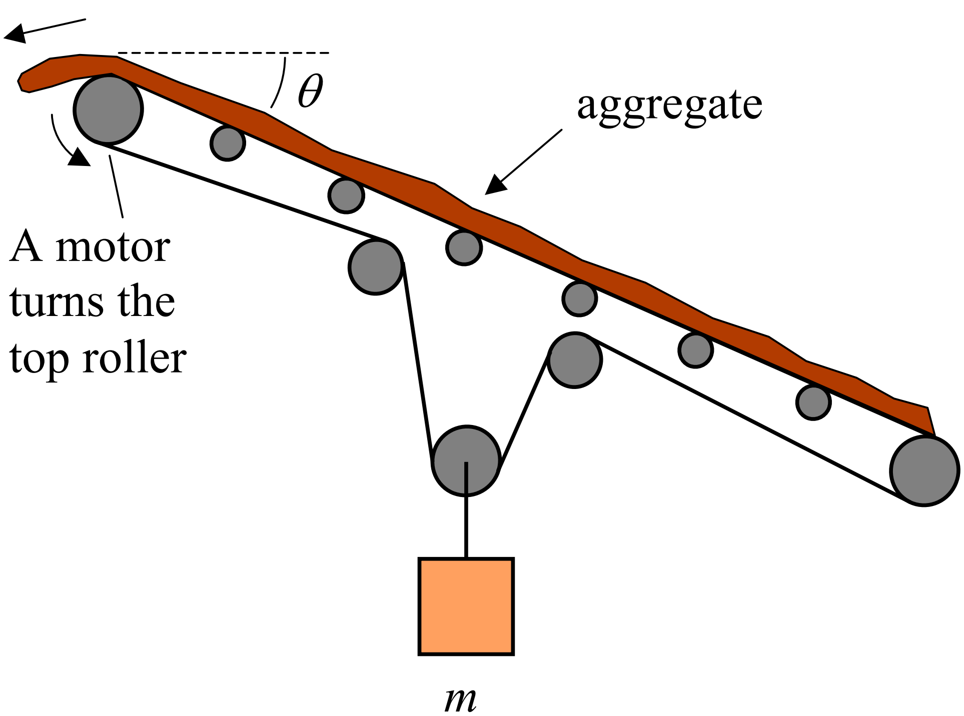
However, the limits of motion of this linkage occurs when angle $\alpha = 0$, which places an upper limit on angle β. This means that the blue linkage of length b cannot rotate around in a full circle. The largest angle of β occurs when $\alpha = 0$.

This particular linkage is known as the Peaucellier-Lipkin linkage, which was invented in 1864. It was the first planar linkage capable of transforming rotational motion into perfect straight-line motion, and vice-versa. This invention became very important for machine components, and for manufacturing.

Problem # 13

A conveyor belt carrying aggregate is illustrated in the figure below. A motor turns the top roller at a constant speed, and the remaining rollers are allowed to spin freely. The belt is inclined at an angle θ. To keep the belt in tension a weight of mass m is suspended from the belt, as shown.

Find the point of maximum tension in the belt. You don't have to calculate it, just find the location and give a reason for it.



Problem # 13 Solution

To determine the point of maximum belt tension, make six imaginary cuts on the belt at the locations shown in the figure below (close to the rollers), and at these locations let T_1, T_2, T_3, T_4, T_5, T_6 represent the (respective) belt tensions.

$T_1 > T_6$ because the belt at location 1 has to support the cable tension T_6 plus a fraction of the belt weight between locations 1 and 6, plus a fraction of the weight of the aggregate between locations 1 and 6. Only a fraction of the weight is supported because the conveyor is inclined at angle θ. This means that only a fraction of the full force of gravity acts in the direction of the belt.

$T_2 > T_3$ because the belt at location 2 has to support the cable tension T_3 plus a fraction of the belt weight that is hanging underneath location 2, and between locations 2 and 3.

$T_3 > T_4$ because there is a longer length of belt hanging below location 3 than there is hanging below location 4.

$T_4 > T_5$ because the belt at location 4 has to support the cable tension T_5 plus a fraction of the belt weight that is hanging underneath location 4, and between locations 4 and 5.

$T_5 \cong T_6$ since locations 5 and 6 are very close to each other and the roller spins freely.

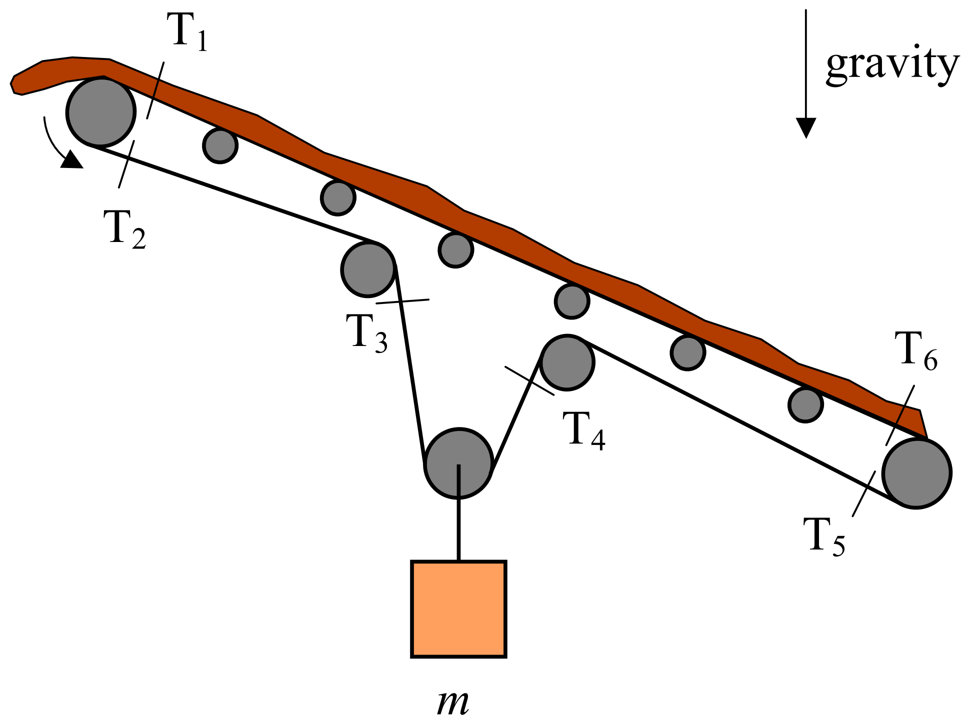
Now, the top roller is turning counter-clockwise at constant rotational speed, which means a counterclockwise motor torque is turning it in this direction. T_1 exerts a clockwise torque on the top roller, and T_2 exerts a counterclockwise torque on the top roller. The roller is turning at constant rotational speed so it is in rotational equilibrium. If we define a sign convention where counterclockwise is positive and clockwise is negative we can write:

(motor torque) + $rT_2 - rT_1 = 0$, where r is the radius of the top roller.

This gives us:

$T_1 = T_2 +$ (motor torque)/r

Therefore, $T_1 > T_2$ and based on the inequalities given above, the maximum belt tension is T_1.

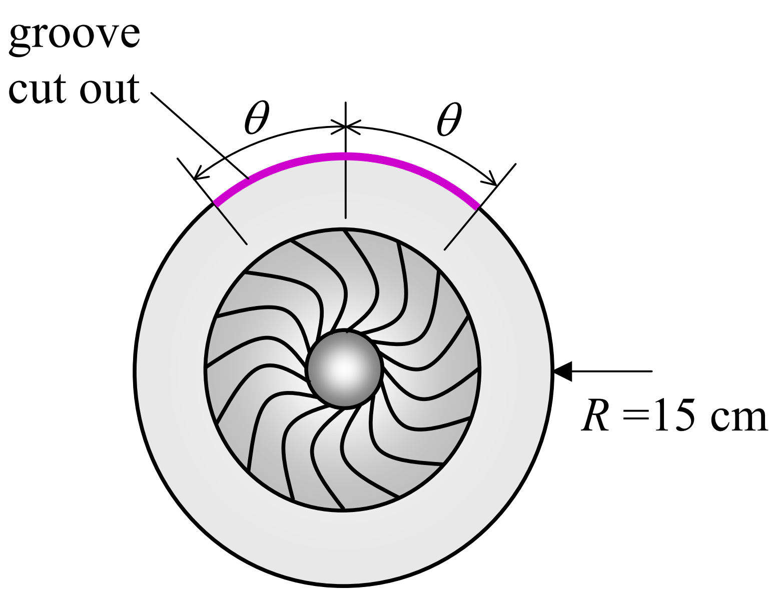

T1
gravity
T2
T3
T4
T6
T5
m

Problem # 14

A quality test has determined that a pump impeller is too heavy on one side by an amount equal to 0.0045 kg·m. To correct this imbalance it is recommended to cut out a groove around the outer circumference of the impeller, using a milling machine, on the same side as the imbalance. This will remove material with the intent of correcting the imbalance. The dimension of the groove is 1 cm wide and 1 cm deep. The groove will be symmetric with respect to the heavy spot. How far around the outer circumference of the impeller should the groove be? Specify the answer in terms of θ. Hint: Treat the groove as a thin ring of material.

The outer radius of the impeller, at the location of the groove, is 15 cm.

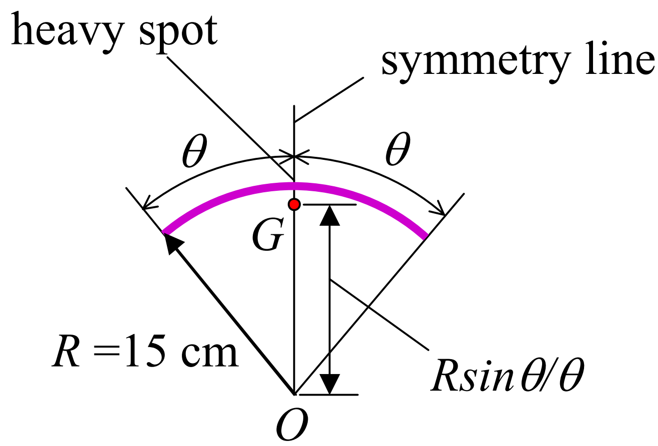
The impeller material is steel, with a density of $\rho = 7900$ kg/m^3.



Problem # 14 Solution

We wish to remove the correct amount of material so that the resulting center of mass of the impeller coincides with the center of rotation of the impeller. If the center of mass does not coincide with the center of rotation, the impeller will vibrate as it rotates.

We shall treat the groove as a thin ring of material. The figure below illustrates the analysis.



The point G represents the center of mass of the material removed from the groove. It is located at a distance $R\sin\theta/\theta$ from the center of rotation O of the impeller (the angle θ is in radians).

The length of the groove is $R(2\theta)$ and the width and depth are both 0.01 m. Therefore, the mass of the material (m_g) removed from the groove is given by

$$m_g = R(2\theta)\times(0.01)\times(0.01)\rho \quad (1)$$

We wish to find the angle θ so that the impeller will be perfectly balanced about the center of rotation O. Mathematically this means that

$$m_g \frac{R\sin\theta}{\theta} = 0.0045$$

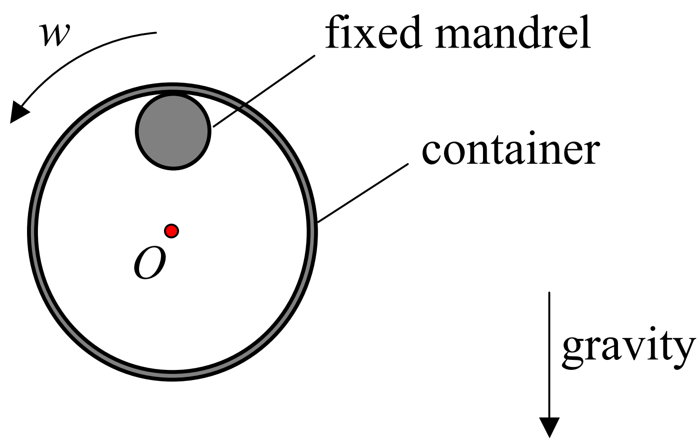
Substituting equation (1) into the above equation we get

$$2R^2 \sin\theta \cdot (0.01)^2 \rho = 0.0045$$

With R = 0.15 m and ρ = 7900 kg/m^3, we solve for θ = 0.127 radians (7.3 degrees).

Problem # 15

As part of a quality check, an axisymmetric container is placed over a very well lubricated fixed mandrel, as shown below. The container is then given an initial pure rotation *w*, with no initial translational motion. What do you expect to see if the center of mass of the container is offset from the geometric center *O* of the container?



Problem # 15 Solution

If the mandrel is very well lubricated we can say that there is no friction force between the container and mandrel. And since there is no friction force, any wobbling motion of the container can only be because the center of mass of the container does not coincide with the geometric center *O* (note: wobbling means that point *O* moves as the container spins). So this is a good quality test for checking the uniformity of the container. Let's look more closely at the reasons for this.

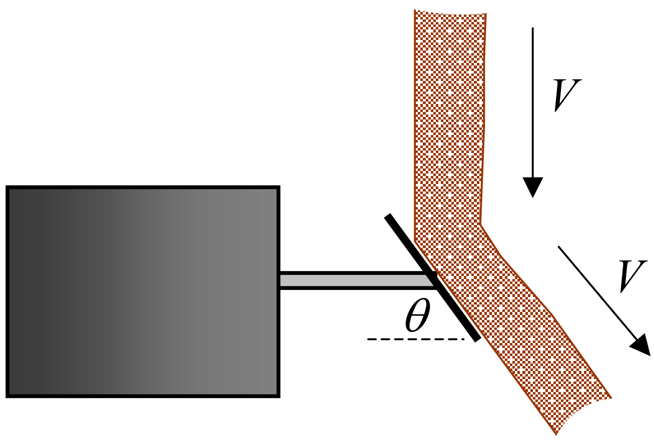
If the center of mass of the container coincides with the geometric center *O*, the container will spin with no wobbling because there is only a normal force (no friction), pushing upward on the container. This normal force exactly balances the force of gravity, and it exerts no moment about *O* because its line of action passes through *O*. Hence, the normal force has no ability to cause wobbling. As a result, no wobbling can only mean that the geometric center *O* coincides with the center of mass of the container. The system will therefore be stable.

However, if the center of mass of the container does <u>not</u> coincide with the geometric center *O*, the container will wobble as it spins. This is because the center of mass of the container is initially set in motion (even though the geometric center of the container *O* has zero initial velocity), and an unbalanced force will result (by Newton's Second Law). As a result, the system will be unstable and the container will wobble.

Problem # 16

A stream of falling material hits the plate of an impact weigher and the horizontal force sensor allows the mass flow rate to be calculated from this. If the speed of the material just before it strikes the plate is equal to the speed of the material just after it strikes the plate, determine an equation for the mass flow rate of the material, based on the horizontal force readout on the sensor. Ignore friction with the plate.

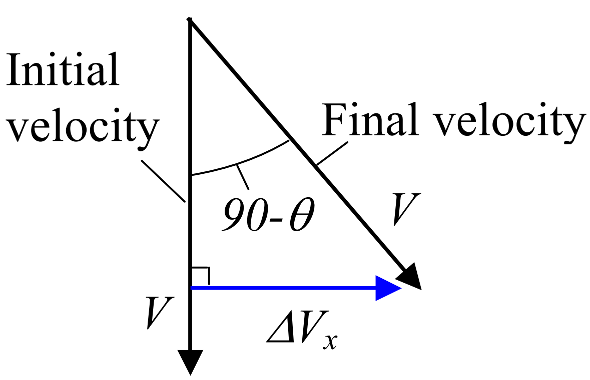
Hint: This can be treated as a fluid flow problem.



Problem # 16 Solution

The velocity of the falling stream of material changes direction as it strikes the plate. This results in acceleration of the material, which results in a force being exerted on the plate.

To determine the horizontal force pushing on the plate, determine the change in horizontal velocity of the material ΔV_x by drawing a vector diagram as shown below.



From geometry, the change in horizontal velocity of the material is given by

$$\Delta V_x = V \sin(90 - \theta) = V \cos\theta$$

Using the general force equation for fluid flow, the horizontal force F_x acting on the plate is

$$F_x = \frac{dm}{dt} \Delta V_x$$

where *dm/dt* is the mass flow rate of the material.

The magnitude of the horizontal force acting on the plate is therefore

$$F_x = \frac{dm}{dt} V \cos\theta$$

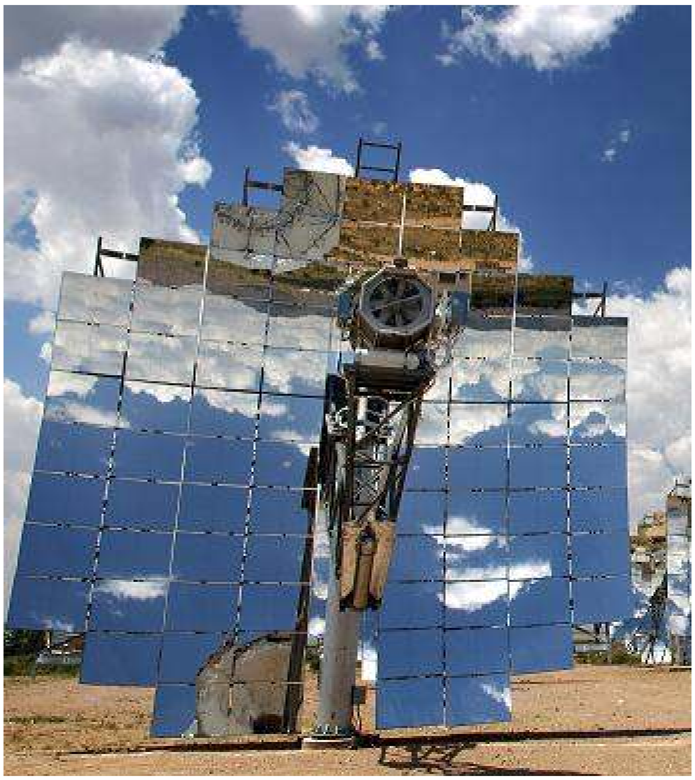
The mass flow rate of the material is therefore given by

$$\frac{dm}{dt} = \frac{F_x}{V \cos\theta}$$

Given the force readout F_x on the sensor, the velocity of the material V, and plate angle θ we can calculate the mass flow rate of the material (from the above equation).

Problem # 17

The SunCatcher is a Stirling engine that is powered by solar energy. It uses large parabolic mirrors to focus sunlight onto a central receiver, which powers a Stirling engine. In the parabolic mirror you can see the reflection of the landscape. Why is the reflection upside down?



Source: http://www.stirlingenergy.com

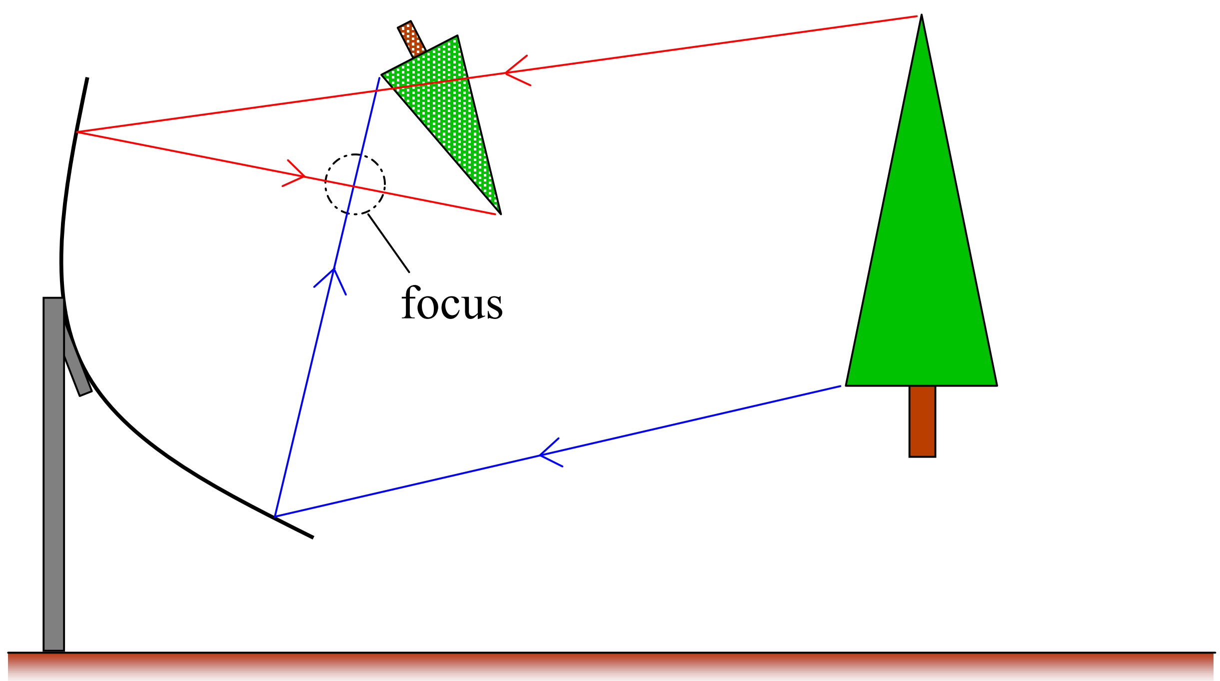
Problem # 17 Solution

To illustrate why the reflection of the landscape is upside down we will draw a basic ray diagram showing how incident light from a tree is reflected off the parabolic dish.

The light hitting the mirrors from the distant landscape (represented by the tree) is reflected towards the mirror focal point where they converge and then diverge (after passing through the focus). This results in the rays of light inverting, which shows the reflection as upside down. So as long as we are standing beyond the focus we will see reflected objects as upside down.

Note: A flat mirror doesn't have a focal point so objects remain upright in their reflection.

If we look at the reflection of the tree on the mirror we see an upside down image of it. However, due to curvature effects of the mirror the image will be somewhat distorted



Problem # 18

On a cold, dry winter day your glasses fog up when you go indoors after being outside for a while. Why is that?

And if you go back outside with your glasses still fogged up, they quickly clear up. Why is that?

Problem # 18 Solution

When you go indoors after being outside, your glasses are cold. This causes the water vapor in the indoor air (which tends to be more humid than outside) to condense on your glasses, causing them to fog up.

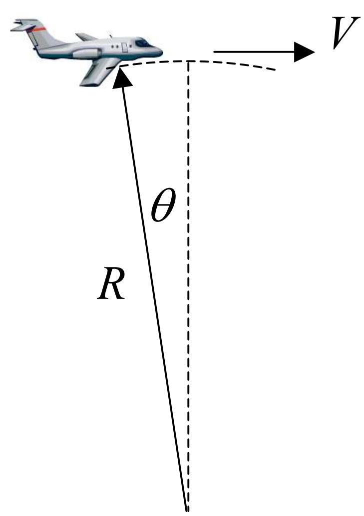
If you go back outside while your glasses are still fogged up, the condensed water quickly evaporates because the dry outdoor air has lower relative humidity than indoors. The dry outdoor air tends to evaporate water quickly, even if it's cold.

Problem # 19

In an astronaut training exercise, an airplane at high altitude travels along a circular arc in order to simulate weightlessness for its passengers. Explain how this is possible.

Problem # 19 Solution

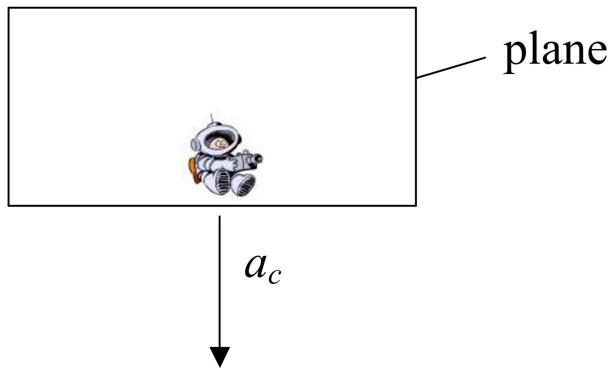
To explain how weightlessness is possible we first illustrate the motion of the airplane in the schematic below.



where V is the velocity of the airplane, R is the radius of the arc circular traveled, and θ is the arc angle as shown.

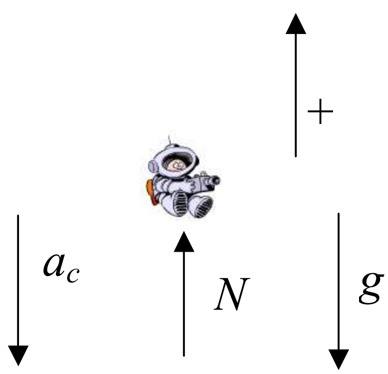
To simulate weightlessness, the centripetal acceleration of the airplane must be equal to the acceleration due to gravity. In addition, the centripetal acceleration of the airplane must be pointing in roughly the same direction as the direction of gravity (downwards). To accomplish this the angle θ must be small.

To show the forces acting on the astronaut, consider the following diagrams.



where a_c is the centripetal acceleration of the airplane, which points downward, towards the center of the arc of radius R.

A free-body diagram of the astronaut (and sign convention) is illustrated as follows.



Where:

N is the contact force between the floor of the airplane and the astronaut

g is the acceleration due to gravity, which is 9.8 m/s^2 near the surface of the earth

Apply Newton's Second law to the astronaut in the vertical direction:

$$N - mg = m(-a_c)$$

where m is the mass of the astronaut.

If $a_c = g$ then $N = 0$, and the astronaut can float (appear weightless) inside the airplane since there is no contact force between him and the floor of the airplane.

The centripetal acceleration a_c of the airplane is given by

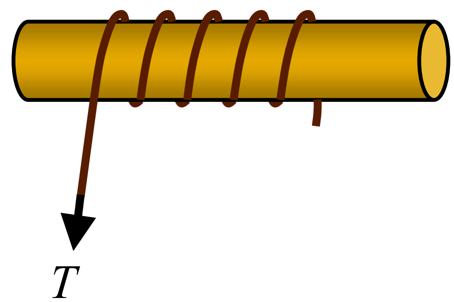
$$a_c = \frac{V^2}{R}$$

For example, if the airplane is flying at 1000 km/h (278 m/s) and R = 7870 m (7.87 km), then a_c = 9.8 m/s^2 and the passengers will feel weightless.

Problem # 20

A rope is wrapped around a pole of radius R = 3 cm. If the tension on one end of the rope is T = 1000 N, and the coefficient of static friction between the rope and pole is μ = 0.2, what is the minimum number of times the rope must be wrapped around the pole so that it doesn't slip off?

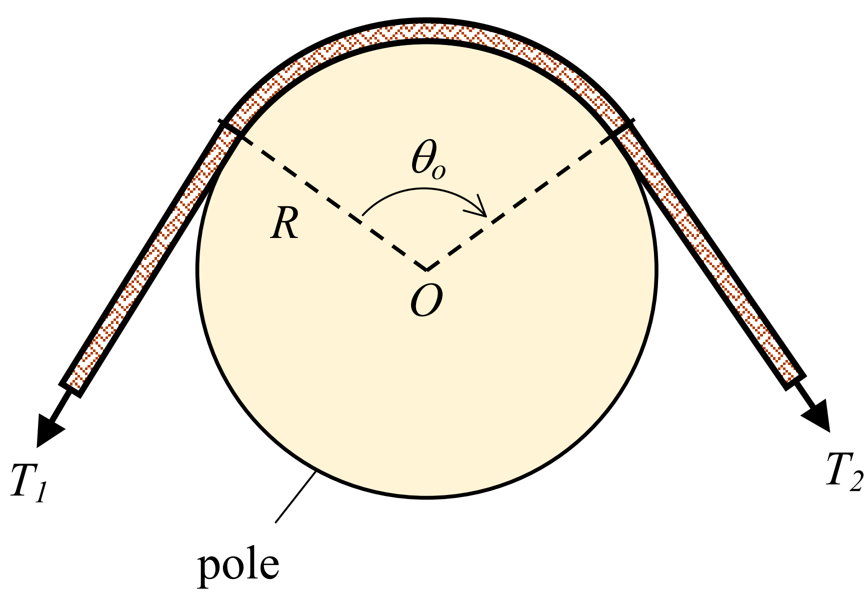
Assume that the minimum number of times the rope must be wrapped around the pole corresponds to a tension of 1 N on the other end of the rope.



Problem # 20 Solution

To solve this problem we have to derive an expression for the rope tension around the pole, using Calculus.

To start, consider a general case as illustrated below.



Where:

T_1 and T_2 is the rope tension on both ends of the rope

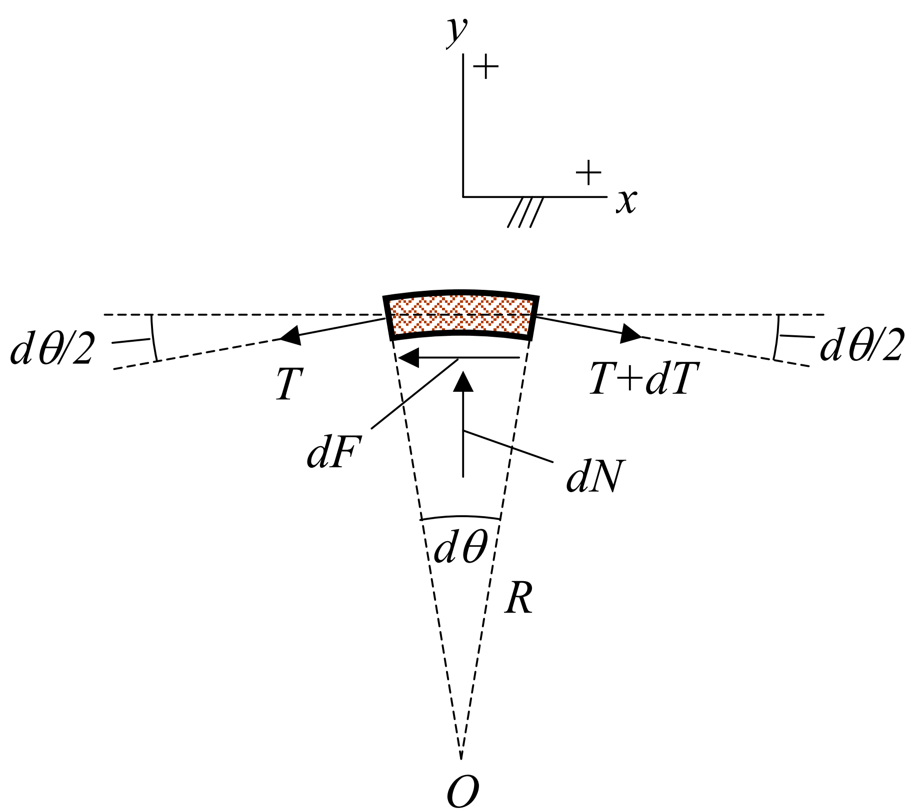
θ_o is the angle the rope is wrapped around the pole, as shown (in radians)

R is the radius of the pole

O is the center of the pole

Next, derive a general expression for the rope tension as a function of T_1, T_2, θ_o, and R.

Consider a differential segment of rope, illustrated below. Treat this as a two-dimensional problem in the *xy* plane.



Where:

dN is the differential normal force between the pole and differential rope segment

dF is the differential friction force between the pole and differential rope segment

T is the rope tension

$d\theta$ is the differential angle spanned by the differential section of rope (in radians)

Since the differential rope segment is in static equilibrium, the sum of the forces acting on it in the xy plane is equal to zero.

In the x-direction, take the sum of the horizontal forces and equate them to zero:

$$(T + dT)\cos\left(\frac{d\theta}{2}\right) - T\cos\left(\frac{d\theta}{2}\right) - dF = 0 \qquad (1)$$

where

$$dF = \mu dN \qquad (2)$$

Similarly, in the y-direction, take the sum of the vertical forces and equate them to zero:

$$dN - T\sin\left(\frac{d\theta}{2}\right) - (T + dT)\sin\left(\frac{d\theta}{2}\right) = 0 \qquad (3)$$

Combine equations (1), (2), (3) and take the limit as $d\theta \to 0$. This gives us

$$\frac{dT}{d\theta} = \mu T$$

We can rewrite this as

$$\frac{dT}{T} = \mu d\theta$$

Integrate both sides of this equation and solve for T as a function of θ. We get

$$T = Ce^{\mu\theta}$$

where C is a constant.

At $\theta = 0$, $T = T_1$, which means that $C = T_1$.

Thus,

$$T_2 = T_1 e^{\mu\theta}$$

It is interesting that this equation does not depend on the radius R of the pole. But this is perhaps not too surprising since R does not show up in equations (1), (2), (3).

Set $\theta = \theta_o$ in order to remain consistent with the variables shown in the figure on page 63.

Therefore,

$$T_2 = T_1 e^{\mu\theta_o}$$

If we assume that $T_2 < T_1$ then we must set $\mu < 0$, since the direction of static friction depends on which direction the rope will tend to slide. This in turn depends on the relative magnitude of T_1 and T_2.

Similarly, if we assume that $T_2 > T_1$ then we must set $\mu > 0$.

In our case, assume that $T_1 = 1000$ N, and $T_2 = 1$ N. This means that we must set $\mu = -0.2$. Using the above equation solve for θ_o.

Solving, we get $\theta_o = 34.54$ radians. This is equal to 5.5 turns, which is the minimum number of times the rope must be wrapped around the pole to prevent it from slipping off.