

The Physics Of Running

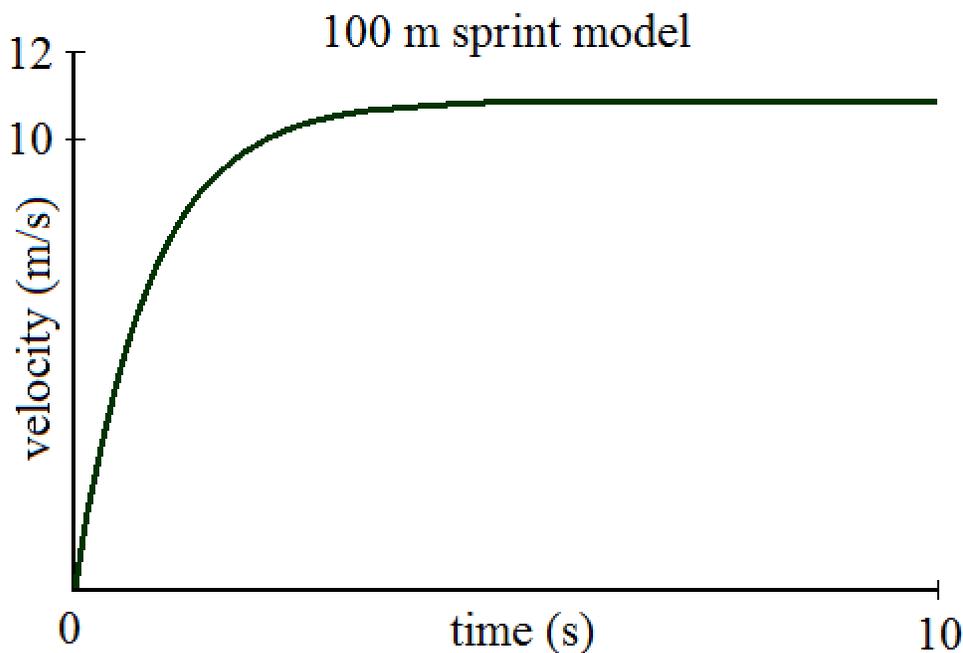
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May 2014

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Case 1 - Optimal running strategy for short sprints less than 291 meters [1, 2]

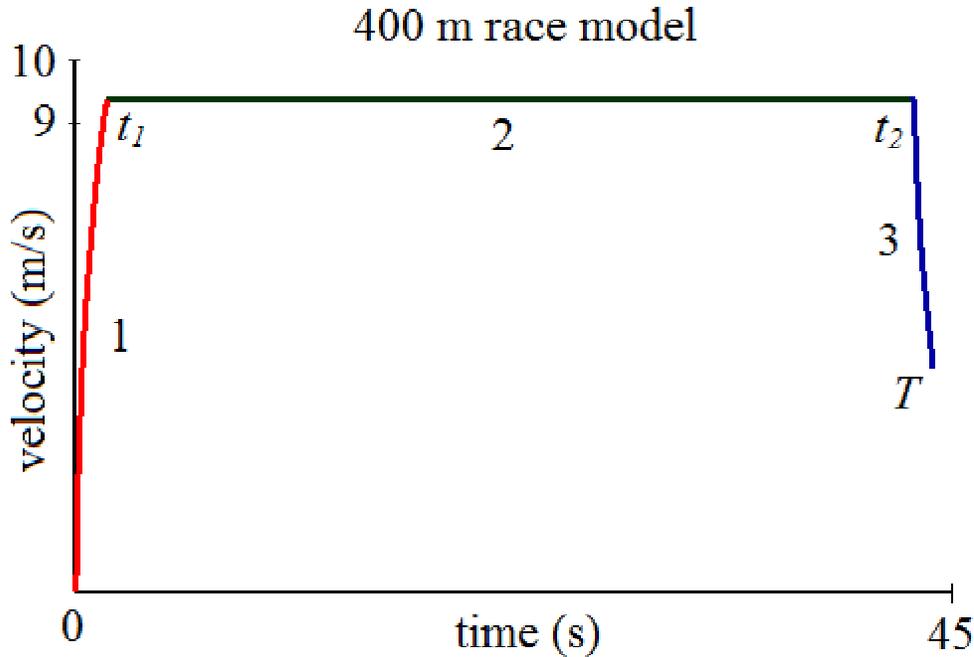
According to an analysis given in reference [2] (and based on the physiology of record holders from 1973), the distance of 291 meters is a transition point in the optimal running strategy. For distances less than 291 meters, the runner should accelerate as fast as possible until he reaches his maximum speed. He should then maintain this speed until the end of the race. The figure below illustrates this for a typical 100 meter men's sprint:



Case 2 - Optimal running strategy for a race longer than 291 meters: The 400 meters [1, 2]

According to an analysis given in reference [2] (and based on the physiology of record holders from 1973), the runner should accelerate as fast as possible for 1.78 seconds. This will enable him to reach a speed near his maximum. He should then maintain this speed for as long as he can. This speed will be such that 0.86 seconds before the end of the race his energy is entirely used up, and after this point is reached his running speed will begin

to drop. Clearly this would be difficult to exactly reproduce in an actual race but it does give some non-intuitive insight into how 400 meter sprinters might maximize their performance. The figure below illustrates this for a typical 400 meter men's race:



Where t_1 and t_2 are intermediate times representing the start and end of the cruising (constant speed) stage. T is the time at which the race ends. Now, $t_1 = 1.78$ seconds, and $t_2 = T - 0.86$ seconds. This is based on the physiology of record holders from 1973.

The numbers 1, 2, 3 represent the three stages of the race: The acceleration stage (1 - the red curve), the cruising stage (2 - the green line), and the deceleration stage (3 - the blue curve).

Mathematical equations for Case 1 and 2 [1, 2]

The optimal running strategies for case 1 and 2 are a result of a mathematical optimization subject to a set of equations which govern the motion of the runner. These equations will be described next.

The length of a race D can be related to the velocity v during the race and the duration T of the race by the following integral equation:

$$D = \int_0^T v(t) dt \quad (1)$$

where t is time.

By Newton's second law,

$$F = ma$$

Where F is the net force, m is the mass, and a is the acceleration. Since $a = dv(t)/dt$ we can write:

$$F = m \frac{dv(t)}{dt}$$

We can apply this equation to the runner where the net force F is the propulsive force, exerted by the runner on the ground, minus the resistive force acting on the runner; which is a combination of internal and external resistance (e.g. internal body friction and external air resistance). It is assumed that this resistance is a linear function of v .

Hence, the above equation becomes

$$f(t)m - \left(\frac{v}{\tau}\right)m = m \frac{dv(t)}{dt}$$

where the left side of the above equation is the net force F . The first term on the left side contains $f(t)$, which is the propulsive force per unit mass of the runner (which is m). The second term on the left side contains (v/τ) , which is the resistive force per unit mass of the runner. The term τ is a constant based on the runner's physiology.

The runner's mass m cancels out of the above equation so that

$$f(t) - \left(\frac{v}{\tau}\right) = \frac{dv(t)}{dt} \quad (2)$$

The above equation is a differential equation in terms of time, which requires that we have an initial condition. The initial condition is:

$$v(0) = 0 \quad (3)$$

which means that the runner starts from rest.

There is a maximum force F_m per unit mass that the runner can exert, which means that

$$f(t) \leq F_m \quad (4)$$

Next, we need an equation for energy since energy is what allows the runner to run and it is something that is used up during the course of the run, much the same way gasoline is used up as a car is driven.

The equation for energy is:

$$\frac{dE(t)}{dt} = \sigma - f(t)v(t) \quad (5)$$

Where:

$E(t)$ is the energy equivalent of the available oxygen per unit mass at time t .

σ is the energy equivalent of the rate at which oxygen is supplied per unit mass in excess of the non-running metabolism. This is a constant based on the runner's physiology.

The term $f(t)v(t)$ is a power term, per unit mass. It is the rate at which the body supplies energy for running.

For the above differential equation (5) we have the initial condition:

$$E(0) = E_o \quad (6)$$

where E_o is the initial energy per unit mass at time $t = 0$. This is based on the runner's physiology.

And finally

$$E(t) \geq 0 \quad (7)$$

The task is now to find the functions $v(t)$, $f(t)$, and $E(t)$ which satisfy equations (2)-(7), such that T , given in equation (1), is minimized. This is therefore an optimization problem requiring the calculus of variations to solve. The details of this solution is not shown here, but those interested in it should see reference [3]. However, the solution for this problem is handily illustrated in the graphs for case 1 and 2 given above.

The equations for case 1 and 2 are given below.

Case 1 equations - optimal for short sprints less than 291 meters [1, 2]

$$f(t) = F_m \quad (\text{for all time } t \text{ during the race of duration } T)$$

$$v(t) = F_m \tau (1 - e^{-t/\tau})$$

$$D = F_m \tau^2 \left(\frac{T}{\tau} + e^{-T/\tau} - 1 \right)$$

Case 2 equations - optimal for races greater than 291 meters and up to 400 meters [1, 2]

Stage 1 (acceleration stage):

$$v(t) = F_m \tau (1 - e^{-t/\tau}) \quad (\text{for } 0 \leq t \leq t_1)$$

Stage 2 (cruising stage):

$$v(t) = \text{constant} \quad (\text{for } t_1 \leq t \leq t_2)$$

Stage 3 (deceleration stage):

$$v(t) = \sqrt{\sigma \tau + [v^2(t_2) - \sigma \tau] e^{2(t_2 - t)/\tau}} \quad (\text{for } t_2 \leq t \leq T)$$

$$E(t) = 0$$

For the above equations for case 1 and 2, we have the following physiological constants, as given in reference [2], for record holders from 1973:

$$\tau = 0.892 \text{ s}$$

$$F_m = 12.2 \text{ m/s}^2$$

$$\sigma = 9.93 \text{ calories}/(\text{kg} \cdot \text{s})$$

$$E_o = 575 \text{ calories/kg}$$

Races greater than 400 meters

For races that are greater than 400 meters the strategy to use is less clear [1].

According to reference [1], "There is typically an acceleration in the later part of the race, either quickly and forcefully to defeat competitors, or gradually over the final third of the race to expend all remaining energy more evenly."

In the absence of a good physical model to assist runners in their racing strategy, they must rely on "feel"; which is a combination of psychological factors based upon their positioning relative to other runners, and how their body feels as they settle into the race.

Predicting long-distance running records for distances of 2000 meters or more [4]

The average velocity during the run is given by

$$V = \frac{L}{T} \quad (8)$$

Where:

V is the average velocity

L is the running distance

T is the duration of the run

The author in reference [4] presents a power law equation, given by:

$$V = \frac{C}{T^s}$$

This is an equation for average velocity V in terms of the race duration T . We have to solve for C given some known V for some T . If we conveniently choose $V = V_1$ at $T = 1$ second then the above equation becomes:

$$V = \frac{V_1}{T^s}$$

If we combine the above equation with equation (8) we have

$$L = \frac{V_1}{T^{s-1}}$$

Hence,

$$T = \left(\frac{V_1}{L} \right)^{1/(s-1)}$$

We can use this equation to solve for T given L .

From reference [4] we have $s = 0.069$ and $V_1 = 10.31$ m/s, which very closely predicts long distance track records for men (as of 1983, the time period of reference [4]). The distances range from 2 km to 42.195 km (marathon). The relative percent error difference between the above equation predictions and the actual records range from 0.05 to 0.7 percent.

The above equation can also be used to predict modern day records, perhaps with a slight adjustment of s and V_1 .

Effect of race curve on running times [5]

Sprinting around a turn (such as for the 200 m and 400 m races) is known to result in a longer race time than if the race were run on a straight track. This is due to the centrifugal force experienced by the runner as he goes around the turn. This has the effect of diminishing the force available to the runner for propelling himself around the track. As expected, this effect is more pronounced the smaller the turn radius is; hence we have the common complaint by runners that the inside lane is "too tight".

Using mathematics, we can estimate how much slower a race is when run around a curved track.

To start, consider first equation (2) from before:

$$f(t) - \left(\frac{v}{\tau} \right) = \frac{dv(t)}{dt}$$

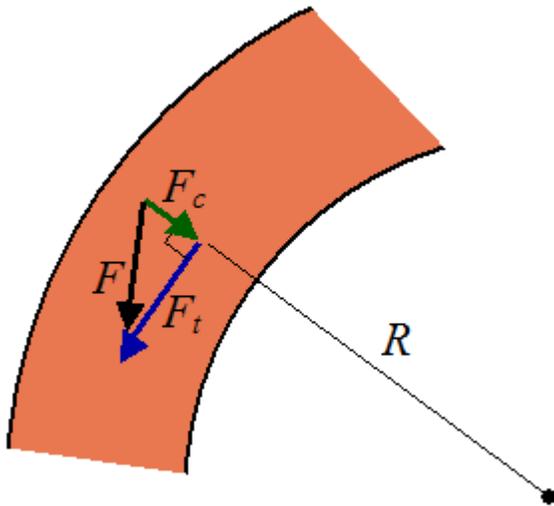
We can treat $f(t)$ as a constant propulsive force, per unit mass of the athlete, so that $f(t) = f$.

This above equation can be written as

$$\frac{dv(t)}{dt} + \left(\frac{v}{\tau} \right) = f$$

This equation readily applies to straight-line motion where the runner is running along a straight track, and f is pointing in the direction of motion of the runner.

Consider the figure below showing a turn radius of R around a curved track. The force F exerted by the runner on the curved track must be such that it pushes him along the track in a tangential direction with force F_t and in addition provides the centrifugal force F_c necessary to maintain his curved running path around the track. Hence the force F has a tangential component F_t and a centrifugal component F_c .



We now wish to apply the above equation for motion around the track, in the tangential direction. Hence, $v(t)$ is in the tangential direction. This means that we must account for the centrifugal force experienced by the runner.

Note that when running around a curved track, f (which is equal to F/m) does not act in the tangential direction. This is due to the presence of F_c .

Hence, to apply the above equation in the tangential direction we must replace the right side with F_t/m . This gives us

$$\frac{dv(t)}{dt} + \left(\frac{v}{\tau}\right) = \frac{F_t}{m}$$

Now, by the Pythagorean theorem,

$$\frac{F_t}{m} = \sqrt{\left(\frac{F}{m}\right)^2 - \left(\frac{F_c}{m}\right)^2}$$

where $F/m = f$ and $F_c/m = v^2/R$ (centripetal acceleration)

Hence,

$$\frac{dv(t)}{dt} + \left(\frac{v}{\tau}\right) = \sqrt{\left(\frac{F}{m}\right)^2 - \left(\frac{F_c}{m}\right)^2}$$

and finally

$$\frac{dv(t)}{dt} + \left(\frac{v}{\tau}\right) = \sqrt{f^2 - \left(\frac{v^2}{R}\right)^2} \quad (9)$$

Let's consider a 200 m race. The starts are staggered so that each runner runs the first 100 m of the race on the curve. However, the radius of curvature of the curved portion of each lane is different. The lanes are 1.22 m wide and the inner radius of the n th lane is given by

$$R(n) = \frac{100}{\pi} + 1.22 \times (n - 1) \text{ meters}$$

Equation (9) cannot be solved analytically. It can only be solved numerically. In reference [5], the authors (solving this equation) determine that a runner in lane 1 (the innermost lane) would run the 200 m in 19.72 seconds, whereas in lane 8 (the outermost lane) he would run it in 19.60 seconds. This is a big time difference and translates into a distance of about 1 meter at the finish line. Since many races are won or lost by mere centimeters, this is a very significant difference. However, it should be pointed out that

sprinters do not consider lane 8 to be the most advantageous one either since, due to the staggered starts, they would run half the race ahead of the other runners which puts them in a psychological disadvantage of not being able to see what the other runners are doing.

For equation (9), we have the following physiological constants for different elite athletes, taken from reference [5]. They are from 1980, the time period of reference [5].

John Carlos ($\tau = 1.5$ s, $f = 8.13$ N/kg)
Bill Gaines ($\tau = 0.8$ s, $f = 13.45$ N/kg)
Jim Hines ($\tau = 1.72$ s, $f = 7.10$ N/kg)
Tommie Smith ($\tau = 0.8$ s, $f = 13.46$ N/kg)

References

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