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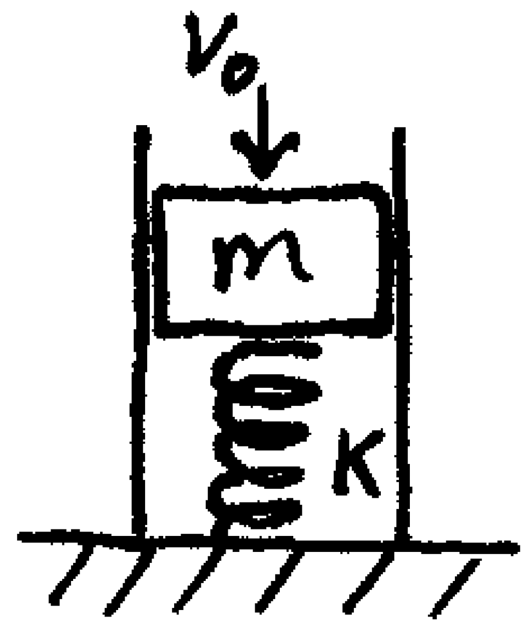
Nov. 12, 2020

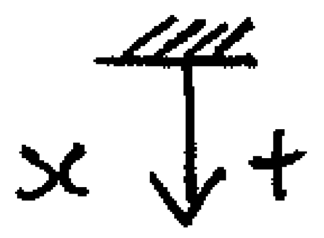


This is a problem involving momentum.




A mass m falling downward contacts a spring of stiffness K, with a velocity $v_o$. What is the duration of the impact (the time that the spring and mass are in contact)? What is the impulse during the impact? What is the average force acting on the mass during the impact?

<u>Solution</u>:




Let x be the amount the spring is compressed. The spring is initially uncompressed.

The force acting on the mass is:

$$\Sigma F_x = \underbrace{-Kx}_{\text{spring force}} + \underbrace{mg}_{\text{gravitational force}}$$

The acceleration of the mass is:

$$a = \frac{d^2x}{dt^2}$$

Apply Newton's second law to the mass:

$$-Kx + mg = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + kx = mg$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = g \qquad (1)$$

The general solution to this is:

$$x(t) = A\cos(wt + B) + C \qquad (2)$$

where A, B, and C are constants, and $w = \sqrt{\frac{k}{m}}$.

substitute equation (2) into (1):

$$\Rightarrow -Aw^2\cos(wt + B) + \left(\frac{k}{m}\right)(A\cos(wt + B) + c) = g$$

$$\Rightarrow \frac{k}{m}\cdot C = g$$

$$C = \frac{mg}{k}$$

A and B must be solved for from the initial conditions:

$$x(0) = 0 \quad (3)$$

$$\frac{dx(0)}{dt} = V_0 \quad (4)$$

From (3), $x(0) = A\cos(B) + \frac{mg}{k} = 0$

From (4), $\frac{dx(0)}{dt} = -Aw\sin(B) = V_0$

$(3) \Rightarrow A\cos(B) = -\dfrac{mg}{K}$

$(4) \Rightarrow A\sin(B) = -\dfrac{V_0}{w}$

Combine (3) and (4):

$$\Rightarrow A^2 = \left(\frac{mg}{K}\right)^2 + \left(\frac{V_0}{w}\right)^2$$

$$A = \sqrt{\left(\frac{mg}{K}\right)^2 + \left(\frac{V_0}{w}\right)^2}$$

and $\tan B = \dfrac{-\frac{V_0}{w}}{-\frac{mg}{K}} = \dfrac{KV_0}{wmg}$

$$\underbrace{B}_{\text{third quadrant angle}} = \tan^{-1}\left(\frac{KV_0}{wmg}\right) + \pi \text{ radians}$$

Therefore,

$$x(t) = \sqrt{\left(\frac{mg}{K}\right)^2 + \left(\frac{V_0}{w}\right)^2} \cos\left( wt + \tan^{-1}\left(\frac{KV_0}{wmg}\right) + \pi\right) + \frac{mg}{K}$$

$$w = \sqrt{\frac{K}{m}}$$

The impact event ends when the mass stops contacting the spring, when the mass is moving upward at velocity $v_0$. In other words,

this assumes no energy is lost during impact (elastic collision)

$$\frac{dx}{dt} = -v_0$$

$$\Rightarrow -\sqrt{\left(\frac{g}{w}\right)^2 + v_0^2}\,\sin\left(wt + \tan^{-1}\left(\frac{kv_0}{wmg}\right) + \pi\right) = -v_0 \quad (5)$$

$$w = \sqrt{\frac{k}{m}}$$

solve for $t$, and this is the duration of the impact. Since "sine" is a periodic function, solve for smallest value of $t$ in above equation.

The impulse during the impact is:

$v_f = -v_0$

$v_i = v_0$

$$J = m(v_f - v_i) \quad \text{(change in linear momentum of mass)}$$

$$\Rightarrow J = m(-v_0 - v_0) = -2mv_0$$

The average force acting on the mass during the impact is:

$$F_{avg} = \frac{J}{t} = -\frac{2mv_0}{t}$$, where $t$ is solved for from equation (5) above