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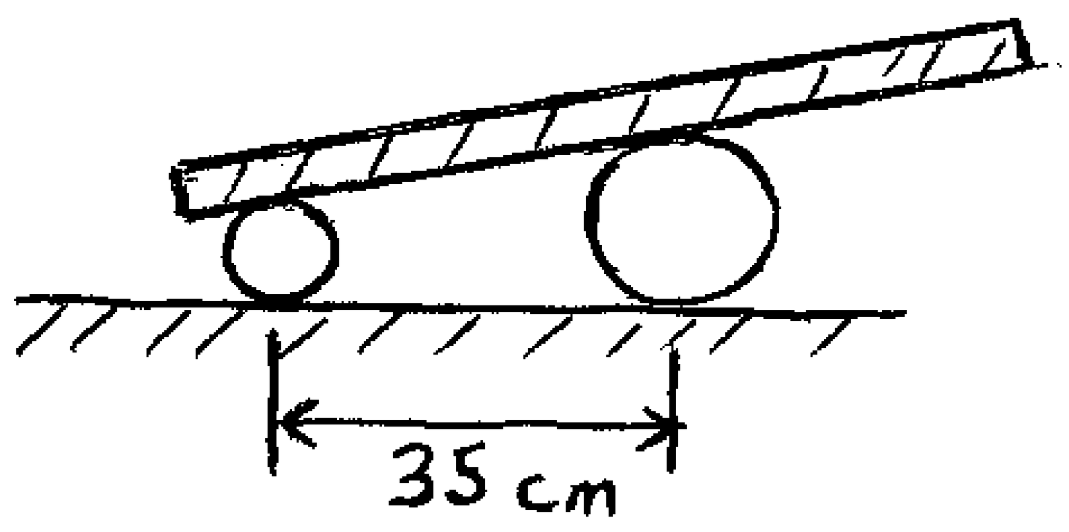
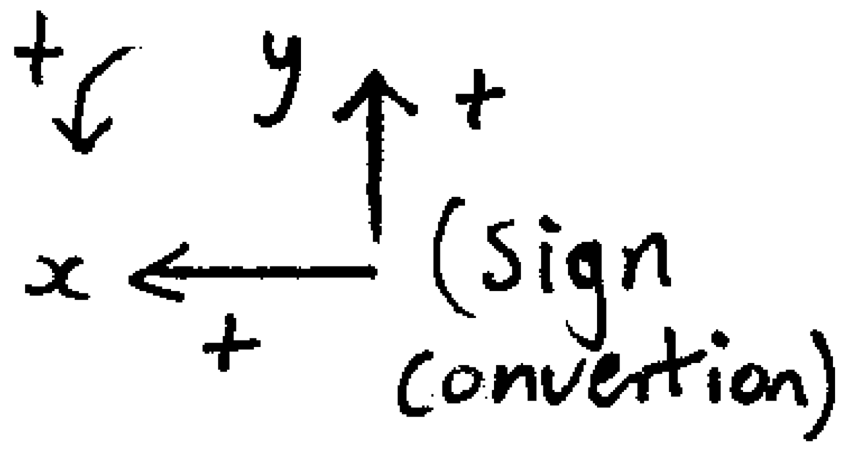
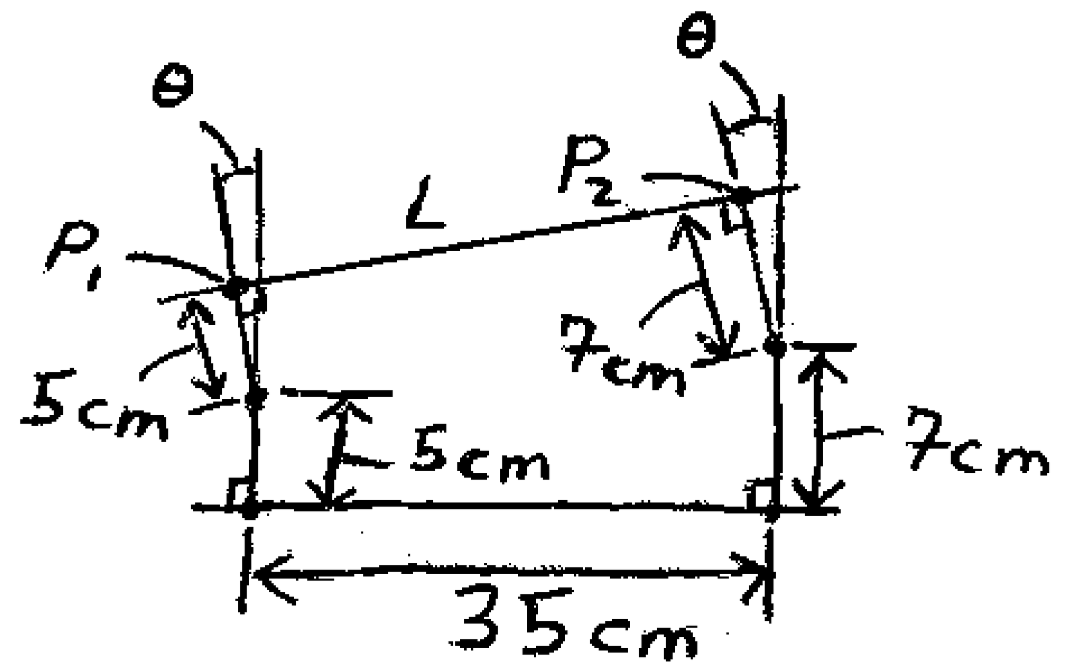
This is a problem involving conservation of energy.



A board is supported by two cylinders on a horizontal surface, as shown. The mass of the board is 13 kg, the mass and radius of the smaller cylinder is 5 kg and 5 cm, and the mass and radius of the larger cylinder is 10 kg and 7 cm. If the system is released from rest and there is no slipping anywhere, what is the velocity of the board after the cylinders have moved a distance of 30 cm? Assume the board stays in contact with the cylinders.

Solution:





The velocity of the center of mass of the smaller cylinder is:

$\vec{V}_{cm_1} = w_1 R_1 \hat{i}$ (this is in the positive x-direction)

- for rolling without slipping,

where w_1, R_1 are the angular velocity and radius of smaller cylinder

P_1 is the contact point between board and smaller cylinder.

P_2 is the contact point between board and larger cylinder.

L is the distance between P_1 and P_2.

Note: $\hat{i}$, $\hat{j}$ are unit vectors pointing along positive x and y directions, respectively

The velocity of the center of mass of the larger cylinder is:

$$\vec{V}_{cm2} = w_2 R_2 \hat{i} \text{ (this is in the positive } x\text{-direction)}$$

- for rolling without slipping, where w_2, R_2 are the angular velocity and radius of larger cylinder

The velocity of point P_1 on the smaller cylinder is:

$$\vec{V}_{P_1} = \vec{V}_{cm1} + w_1 R_1 \cos\theta \, \hat{i} - w_1 R_1 \sin\theta \, \hat{j}$$

$$\Rightarrow \vec{V}_{P_1} = w_1 R_1 (1 + \cos\theta) \hat{i} - w_1 R_1 \sin\theta \, \hat{j} \quad (1)$$

Similarly, the velocity of Point P_2 on the larger cylinder is:

$$\vec{V}_{P_2} = \vec{V}_{cm2} + w_2 R_2 \cos\theta \, \hat{i} - w_2 R_2 \sin\theta \, \hat{j}$$

$$\Rightarrow \vec{V}_{P_2} = w_2 R_2 (1 + \cos\theta) \hat{i} - w_2 R_2 \sin\theta \, \hat{j} \quad (2)$$

The velocity of point P_1 on the board is:

$$\vec{V}_{P_1} = \underbrace{\vec{V}_{P_2}}_{\text{velocity of point } P_2 \text{ on the board}} - w_b L \cos\theta \, \hat{j} - w_b L \sin\theta \, \hat{i} \quad (3)$$

, where w_b is the angular velocity of the board

Note: $\left(w_b = \frac{d\theta}{dt} \right)$

Note that P_1 and P_2 do not move relative to the board because of the no slip condition. If there was slip there would need to be a relative velocity term added to equation (3).

Substitute equations (1) and (2) into equation (3):

$$w_1 R_1 (1+\cos\theta)\hat{i} - w_1 R_1 \sin\theta \hat{j}$$
$$= w_2 R_2 (1+\cos\theta)\hat{i} - w_2 R_2 \sin\theta \hat{j}$$
$$- w_b L \cos\theta \hat{j} - w_b L \sin\theta \hat{i}$$

By comparison:

(A) $w_1 R_1 (1+\cos\theta) = w_2 R_2 (1+\cos\theta) - w_b L \sin\theta$

and

(B) $-w_1 R_1 \sin\theta = -w_2 R_2 \sin\theta - w_b L \cos\theta$

(A) x $\sin\theta$ + (B) x $(1+\cos\theta)$:

$$\Rightarrow \quad 0 = -w_b L \sin^2\theta - w_b L \cos\theta (1+\cos\theta)$$

$$\Rightarrow \quad 0 = -w_b L (1+\cos\theta)$$

The only possible solution is $w_b = 0$, which then means that $w_1 R_1 = w_2 R_2$, from equations (A) and (B).

From equation (3), this means that $\vec{V}_{P_1} = \vec{V}_{P_2}$ (board velocity - since P_1 and P_2 are points on the board which do not move relative to the board)

From equations (1) and (2),

the board velocity is:

$$\vec{V}_{board} = w_1 R_1 (1+\cos\theta)\hat{i} - w_1 R_1 \sin\theta \hat{j}$$

and $w_1 R_1 = w_2 R_2$

Since $w_b = 0$, the board velocity given previously is valid for every point on the board, including its center of mass. The board only experiences translational motion, and no rotational motion.

Now, $|\vec{V}_{board}|^2 = 2(w_1 R_1)^2 (1 + \cos\theta)$ (4)

The velocity of the center of mass of the smaller cylinder is:

$$\vec{V}_{cm_1} = w_1 R_1 \hat{i}$$

so that, $|\vec{V}_{cm_1}|^2 = (w_1 R_1)^2$ (5)

The velocity of the center of mass of the larger cylinder is:

$$\vec{V}_{cm_2} = w_2 R_2 \hat{i}$$

so that, $|\vec{V}_{cm_2}|^2 = (w_2 R_2)^2 = (w_1 R_1)^2$ (6)

The angular velocity of the smaller cylinder is w_1.

The angular velocity of the larger cylinder is: $w_2 = \dfrac{w_1 R_1}{R_2}$ (since $w_1 R_1 = w_2 R_2$)

Since $w_1 R_1 = w_2 R_2$, $\vec{V}_{cm_1} = \vec{V}_{cm_2}$, and as a result, the distance between the center of mass of both cylinders stays constant, at 35 cm.

From the geometry shown on the diagram on the first page (proof not shown),

$$\tan\left(\frac{\theta}{2}\right) = \frac{7-5}{35}, \quad \theta = 6.54°, \quad \theta = \text{constant throughout motion}$$

The cylinders moving a distance of 30 cm means that their center of mass moves 30 cm. Since the cylinders roll on a horizontal surface, their center of mass does not change in elevation, and they don't need to be considered for grav. pot. energy

Now, $\vec{v}_{cm_1} = \vec{v}_{cm_2}$, so rolling distance of center of mass $= \int_0^T w_1 R_1 \, dt = 0.30m$

where T is the time it takes for the cylinders to roll 30 cm

The board velocity in the downward (negative y) direction is $w_1 R_1 \sin\theta$

So the change in height of the center of mass of board is:

$$\Delta H = -\int_0^T w_1 R_1 \sin\theta \, dt$$

$$\Rightarrow \Delta H = -\sin\theta \int_0^T w_1 R_1 \, dt = -0.30 \sin\theta \quad \text{(meters)}$$

Lastly, apply conservation of energy equation:

$$T_1 + V_1 = T_2 + V_2$$

$T_1 = 0$ (system starts from rest)

$V_1 = 0$ (datum for center of mass of board chosen at the location of its mass center)

$$T_2 = \overbrace{\frac{1}{2} m_{c_1} |\vec{V}_{cm_1}|^2 + \frac{1}{2} I_{G_1} w_1^2}^{\text{kinetic energy of smaller cylinder}}$$

$$+ \underbrace{\frac{1}{2} m_{c_2} |\vec{V}_{cm_2}|^2 + \frac{1}{2} I_{G_2} w_2^2}_{\text{Kinetic energy of larger cylinder}} + \underbrace{\frac{1}{2} m_b |\vec{V}_{board}|^2}_{\text{Kinetic energy of board}}$$

$$V_2 = m_b g(\Delta H)$$

Substitute above quantities into conservation of energy equation, and also substitute $m_{c_1} = 5\text{ kg}$, $|\vec{V}_{cm_1}|^2 = (w_1 R_1)^2 = w_1^2 (0.05)^2$, $I_{G_1} = \frac{1}{2} m_{c_1} R_1^2 = \frac{1}{2}(5)(0.05)^2$, $m_{c_2} = 10\text{ kg}$, $|\vec{V}_{cm_2}|^2 = (w_1 R_1)^2 = w_1^2 (0.05)^2$, $I_{G_2} = \frac{1}{2} m_{c_2} R_2^2 = \frac{1}{2}(10)(0.07)^2$, $w_2 = w_1(R_1/R_2) = w_1(5/7)$, $m_b = 13\text{ kg}$, $|\vec{V}_{board}|^2 = 2w_1^2 (0.05)^2 (1 + \cos 6.54°)$, $\Delta H = -0.3 \sin 6.54° = -0.03417\text{ m}$

The result is:

$$0 = 0.00625 w_1^2 + 0.003125 w_1^2 + 0.0125 w_1^2$$
$$+ 0.00625 w_1^2 + 0.06479 w_1^2 - 4.353$$

Solve for $w_1 = 6.84$ rad/s

Then, $\vec{V}_{board} = 0.68\,\hat{i} - 0.039\,\hat{j}$ m/s (answer)