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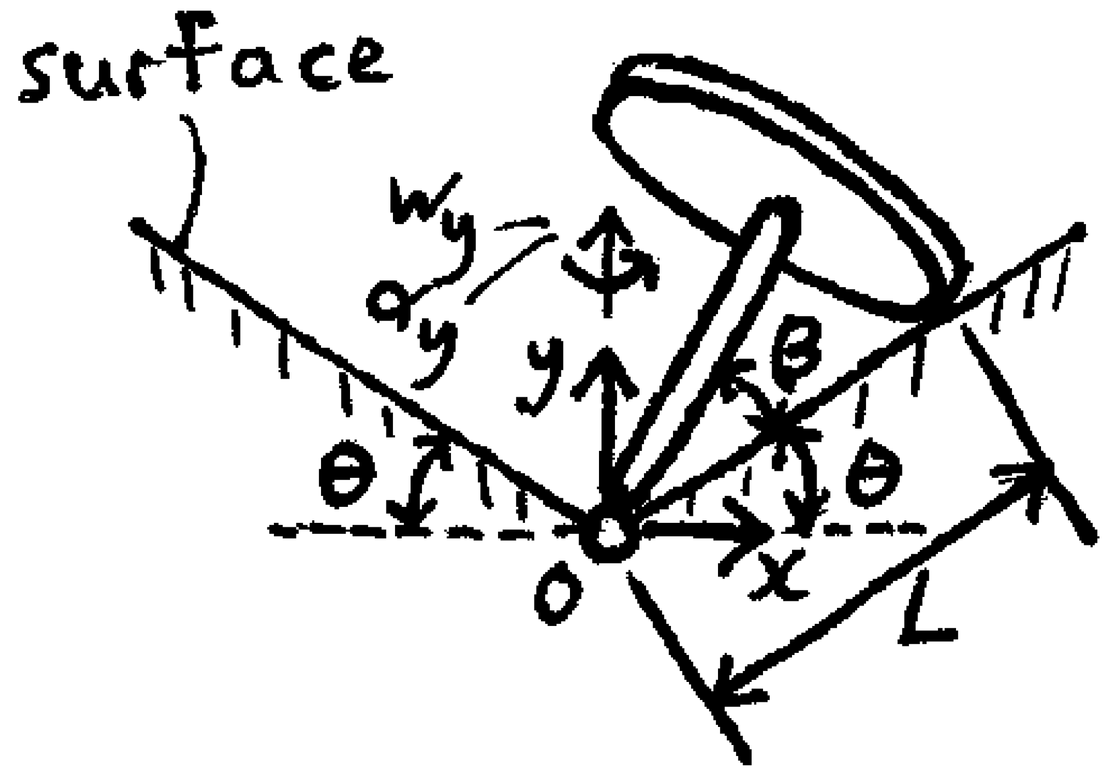
This is a 3D general motion problem (engineering mechanics).



A wheel rolls along a conical surface without slipping. The wheel is connected to a shaft which has a ball-and-socket at O, around which the shaft and wheel rotate. The wheel and shaft precess around point O, with an angular velocity of w_y, and an angular acceleration of α_y. Determine the angular velocity and angular acceleration of the wheel, at the instant shown.

<u>Solution:</u>

Let P be the contact point between the wheel and the surface. Then,

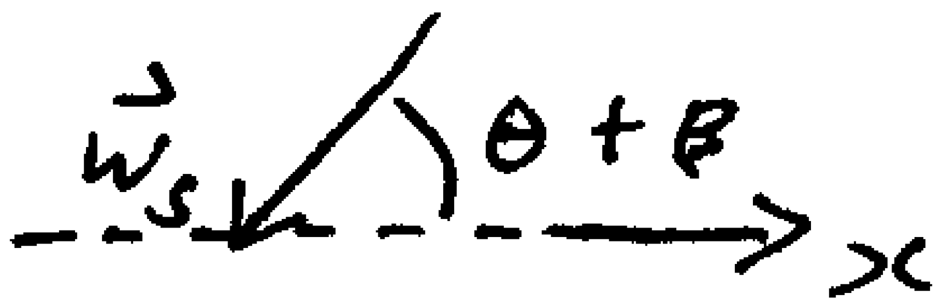
$$\vec{v}_P = \underbrace{\vec{v}_0}_{0} + \vec{w} \times \vec{r}_{P/O} = 0, \quad \text{since the wheel rolls without slipping}$$

Then, $\vec{w} \times \vec{r}_{P/O} = 0$

$$\vec{r}_{P/O} = L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}$$

$$\vec{w} = -w_s \cos(\theta+\beta)\,\hat{i} + \left(-w_s \sin(\theta+\beta) + w_y\right)\hat{j}$$

$\rightarrow$ where w_s is the spin rate of the wheel, unknown right now.



Now,

$$\vec{w} \times \vec{r}_{P/O} = \left(-w_s \cos(\theta+\beta)\,\hat{i} + \left(-w_s \sin(\theta+\beta) + w_y\right)\hat{j}\right) \times \left(L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}\right) = 0$$

$$\Rightarrow \vec{w} \times \vec{r}_{P/O} = -w_s \cos(\theta+\beta) L \sin\theta\,\hat{k} - \left(-w_s \sin(\theta+\beta) + w_y\right) \cdot L\cos\theta\,\hat{k} = 0$$

This is true only if,

$$-w_s \cos(\theta+\beta) L\sin\theta - \left(-w_s \sin(\theta+\beta) + w_y\right) L\cos\theta = 0$$

Solve for $w_s = \dfrac{w_y L\cos\theta}{\sin(\theta+\beta)L\cos\theta - \cos(\theta+\beta)L\sin\theta}$

$$w_s = \frac{w_y \cos\theta}{\sin(\theta+\beta)\cos\theta - \cos(\theta+\beta)\sin\theta}$$

$$w_s = \frac{w_y \cos\theta}{\sin(\theta+\beta-\theta)} = \frac{w_y \cos\theta}{\sin\beta} \text{ rad/s}$$

Therefore, $\vec{w} = -w_y \dfrac{\cos\theta}{\sin\beta} \cos(\theta+\beta)\hat{i}$

$$+ \left(-w_y \frac{\cos\theta}{\sin\beta} \sin(\theta+\beta) + w_y\right)\hat{j}$$

rad/s

(answer)

Next, $\vec{a}_p = \underbrace{\vec{a}_o}_{0} + \vec{\alpha} \times \vec{r}_{P/O} + \vec{w} \times (\vec{w} \times \vec{r}_{P/O})$ (1)

The tangential component of $\vec{a}_p$ (along the z-direction) is 0, due to the no-slip condition.

Now, $\vec{\alpha} = -\alpha_s \cos(\theta+\beta)\hat{i} + (-\alpha_s \sin(\theta+\beta) + \alpha_y)\hat{j}$

$+ \underbrace{\vec{w}_y \times \vec{w}_s}$ → where α_s is the spin angular acceleration of the wheel, unknown right now.

The w_y rotation vector causes the rotation of the w_s vector. Note that the w_y rotation vector always points along y-axis (constant direction).

$\vec{w}_s = -w_s \cos(\theta+\beta)\,\hat{i} - w_s \sin(\theta+\beta)\hat{j}$

$\vec{w}_y = w_y\,\hat{j}$

Substitute known quantities into above equation:

$$\vec{\alpha} = -\alpha_s \cos(\theta+\beta)\hat{i} + (-\alpha_s \sin(\theta+\beta) + \alpha_y)\hat{j}$$
$$+ w_y \hat{j} \times (-w_s \cos(\theta+\beta)\hat{i} - w_s \sin(\theta+\beta)\hat{j})$$

In equation (1), the only term which contains a z-component is $\vec{\alpha} \times \vec{r}_{P/O}$, so we only need to consider this term.

Now,

$$\vec{\alpha} \times \vec{r}_{P/O} = \Big(-\alpha_s \cos(\theta+\beta)\hat{i} + (-\alpha_s \sin(\theta+\beta) + \alpha_y)\hat{j}$$
$$+ w_y \hat{j} \times (-w_s \cos(\theta+\beta)\hat{i} - w_s \sin(\theta+\beta)\hat{j})\Big)$$
$$\times \left(L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}\right)$$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-\alpha_s \cos(\theta+\beta)\hat{i} + (-\alpha_s \sin(\theta+\beta) + \alpha_y)\hat{j}\right.$$
$$\left. + w_y w_s \cos(\theta+\beta)\hat{k}\right) \times \left(L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}\right)$$

From this, the z-component term is:

$$\left[-\alpha_s \cos(\theta+\beta) L\sin\theta - (-\alpha_s \sin(\theta+\beta) + \alpha_y) L\cos\theta\right]\hat{k}$$

The z-component term must be zero, so

$$-\alpha_s \cos(\theta+\beta)\sin\theta + \alpha_s \sin(\theta+\beta)\cos\theta - \alpha_y \cos\theta = 0$$

Solve for $\alpha_s = \dfrac{\alpha_y \cos\theta}{\sin(\theta+\beta)\cos\theta - \cos(\theta+\beta)\sin\theta}$

$$\alpha_s = \frac{\alpha_y \cos\theta}{\sin\beta} \quad \mathrm{rad}/\mathrm{s}^2$$

Therefore,

$$\vec{\alpha} = -\alpha_y \frac{\cos\theta}{\sin\beta}\cos(\theta+\beta)\,\hat{\imath} + \left(-\alpha_y \frac{\cos\theta}{\sin\beta}\sin(\theta+\beta) + \alpha_y\right)\hat{\jmath}$$

$$+\, w_y^2 \frac{\cos\theta}{\sin\beta}\cos(\theta+\beta)\,\hat{k} \quad \mathrm{rad}/\mathrm{s}^2$$

(answer)