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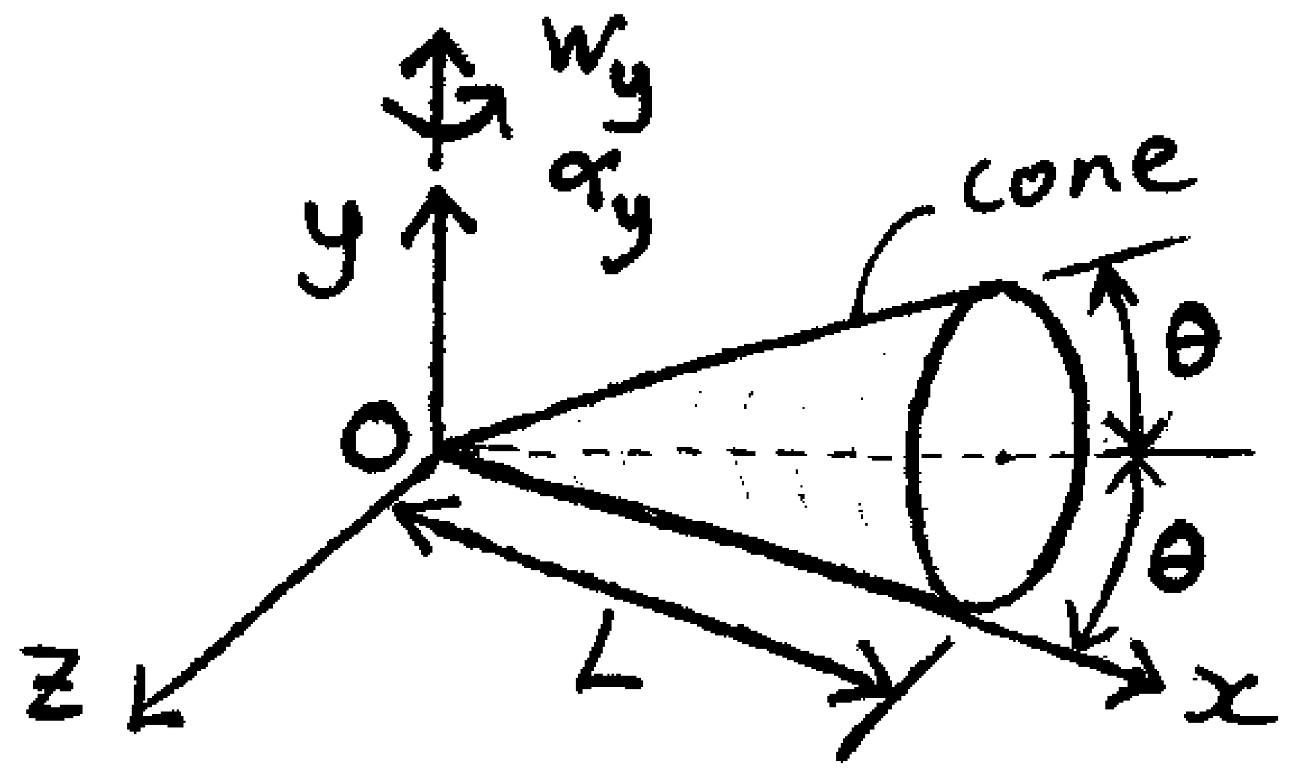
This is a 3D general motion problem (engineering mechanics).



A cone rolls on the xz-plane without slipping, with its tip at the origin O, around which the cone rotates. The cone precesses around point O, with an angular velocity w_y, and an angular acceleration α_y. Determine the angular velocity and angular acceleration of the cone, at the instant shown.

Solution:

Let P be the contact point between the cone and the xz-plane, at the end of the cone, located a distance L from the origin. However, we can use any point of contact along the contact "line" which is along the x-axis for $0 \le x \le L$. Then,

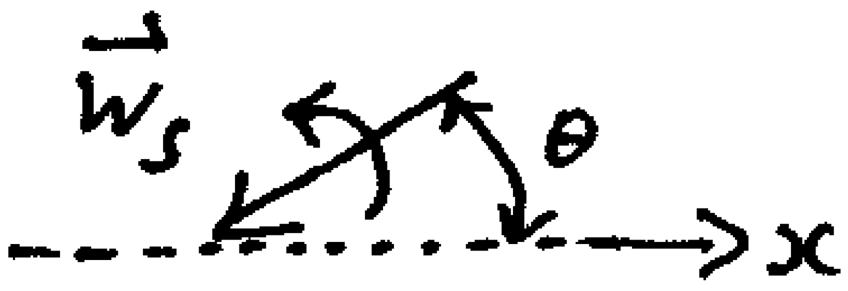
$\vec{V}_P = \vec{V}_0 + \vec{w} \times \vec{r}_{P/0} = 0$, since the cone rolls without slipping

Then, $\vec{w} \times \vec{r}_{P/0} = 0$

$\vec{r}_{P/0} = L\,\hat{i}$

$$\vec{w} = -w_s \cos\theta \, \hat{i} + (-w_s \sin\theta + w_y)\, \hat{j}$$

→ where w_s is the spin rate of the cone, unknown right now.



Now, $\vec{w} \times \vec{r}_{P/O} = (-w_s \cos\theta \, \hat{i} + (-w_s \sin\theta + w_y)\, \hat{j}) \times (L \, \hat{i})$

$= 0$

This is only true if,

$$-w_s \sin\theta + w_y = 0$$

Solve for $w_s = \dfrac{w_y}{\sin\theta}$ rad/s

Therefore, $\vec{w} = -\dfrac{w_y}{\tan\theta} \, \hat{i}$ (answer)

rad/s

Next, $\vec{a}_p = \underbrace{\vec{a}_o}_{0} + \vec{\alpha} \times \vec{r}_{P/O} + \vec{w} \times (\vec{w} \times \vec{r}_{P/O})$ (1)

The tangential component of $\vec{a}_p$ (along the z-direction) is 0, due to the no-slip condition.

Now, $\vec{\alpha} = -\alpha_s \cos\theta\,\hat{i} + (-\alpha_s \sin\theta + \alpha_y)\,\hat{j}$

$+ \underbrace{\vec{w}_y \times \vec{w}_s}$ → where α_s is the spin angular acceleration of the cone, unknown right now.

$\vec{w}_s = -w_s\cos\theta\,\hat{i} - w_s \sin\theta\,\hat{j}$

$\vec{w}_y = w_y\,\hat{j}$

The w_y rotation vector causes the rotation of the w_s vector. Note that the w_y rotation vector always points along y-axis (constant direction).

Substitute known quantities into above equation:

$$\vec{\alpha} = -\alpha_s \cos\theta\,\hat{i} + (-\alpha_s \sin\theta + \alpha_y)\,\hat{j} + w_y\,\hat{j} \times (-w_s \cos\theta\,\hat{i} - w_s \sin\theta\,\hat{j})$$

$$\vec{\alpha} = -\alpha_s \cos\theta\,\hat{i} + (-\alpha_s \sin\theta + \alpha_y)\,\hat{j} + w_y\,\hat{j} \times \left(-\frac{w_y}{\tan\theta}\,\hat{i} - w_y\,\hat{j}\right)$$

In equation (1), the only term which contains a z-component is $\vec{\alpha} \times \vec{r}_{P/O}$, so we only need to consider this term.

Now,

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-\alpha_s \cos\theta\,\hat{i} + (-\alpha_s \sin\theta + \alpha_y)\,\hat{j} + w_y\,\hat{j} \times \left(-\frac{w_y}{\tan\theta}\,\hat{i} - w_y\,\hat{j}\right)\right) \times L\,\hat{i}$$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-\alpha_s \cos\theta\, \hat{i} + (-\alpha_s \sin\theta + \alpha_y)\, \hat{j} + \frac{w_y^2}{\tan\theta}\, \hat{k}\right)$$

$\times L \hat{i}$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left[-(-\alpha_s \sin\theta + \alpha_y)\, \hat{k} + \frac{w_y^2}{\tan\theta}\, \hat{j}\right] L$$

The z-component term must be 0, so

$$-\alpha_s \sin\theta + \alpha_y = 0$$

Solve for $\alpha_s = \dfrac{\alpha_y}{\sin\theta}$ rad/s²

Therefore,

$$\vec{\alpha} = -\frac{\alpha_y}{\tan\theta}\, \hat{i} + \frac{w_y^2}{\tan\theta}\, \hat{k} \quad \text{rad/s}^2 \quad \text{(answer)}$$

Note that L does not enter in the solution, which means that L can be any length and the solution won't change!