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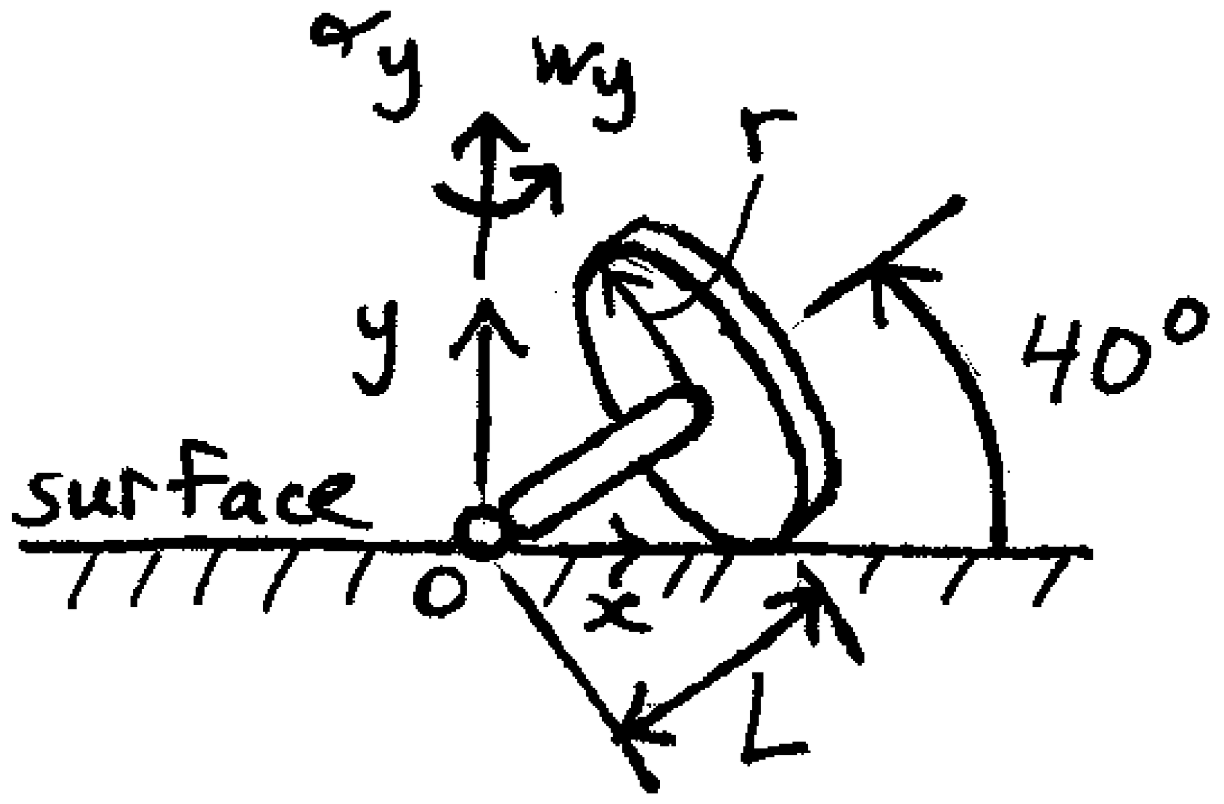
May 13, 2020

This is a 3D general motion problem (engineering mechanics).



A wheel rolls along a level surface without slipping. The wheel is connected to a shaft which has a ball-and-socket joint at O, around which the shaft and wheel rotate. The wheel and shaft precess around point O, with an angular velocity $w_y = 6 \text{ rad/s}$, and an angular acceleration of $\alpha_y = 2 \text{ rad/s}^2$. Determine the angular velocity and angular acceleration of the wheel, at the instant shown.

Solution:

Let P be the contact point between the wheel and the surface. Then,

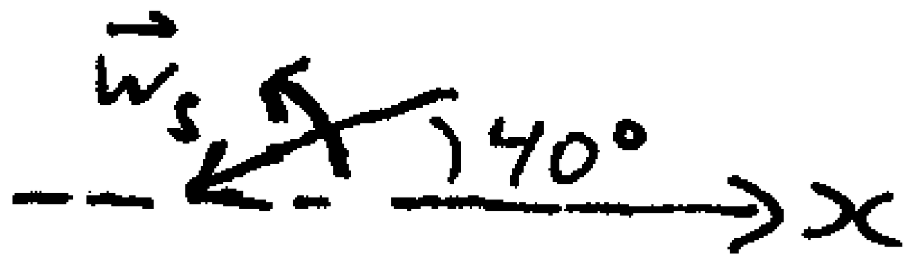
$$\underbrace{\vec{v}_P = \vec{v}_O}_{0} + \vec{w} \times \vec{r}_{P/O} = 0$$, since the wheel rolls without slipping

Then, $\vec{w} \times \vec{r}_{P/O} = 0$

$$\vec{r}_{P/O} = \frac{L}{\cos 40^\circ}\,\hat{i}$$

$$\vec{w} = -w_s \cos 40° \hat{i} + (-w_s \sin 40° + w_y)\hat{j}$$

→ where w_s is the spin rate of the wheel, unknown right now.



Now,

$$\vec{w} \times \vec{r}_{P/O} = (-w_s \cos 40° \hat{i} + (-w_s \sin 40° + w_y)\hat{j}) \times \left(\frac{L}{\cos 40°} \hat{i}\right) = 0$$

This is only true if,

$$-w_s \sin 40° + w_y = 0$$

Solve $w_s = \dfrac{w_y}{\sin 40°} = \dfrac{6}{\sin 40°}$ rad/s

Therefore, $\vec{w} = \dfrac{-6}{\sin 40°} \cos 40° \hat{i} = \underline{\underline{-7.15 \hat{i}}}$ rad/s (answer)

Next, $\vec{a}_p = \underbrace{\vec{a}_o}_{0} + \vec{\alpha} \times \vec{r}_{P/O} + \vec{w} \times (\vec{w} \times \vec{r}_{P/O})$ (1)

The tangential component of $\vec{a}_p$ (along the z-direction) is 0, due to the no-slip condition.

Now, $\vec{\alpha} = -\alpha_s \cos 40° \hat{i} + (-\alpha_s \sin 40° + \alpha_y)\hat{j}$

$+ \underbrace{\vec{w}_y \times \vec{w}_s}$ → where α_s is the spin angular acceleration of the wheel, unknown right now.

$\vec{w}_s = -w_s \cos 40° \hat{i} - w_s \sin 40° \hat{j}$

$\vec{w}_y = 6\hat{j}$

The w_y rotation vector causes the rotation of the w_s vector. Note that the w_y rotation vector always points along y-axis (constant direction).

Substitute known quantities into above equation:

$$\vec{\alpha} = -\alpha_s \cos 40° \hat{i} + (-\alpha_s \sin 40° + 2)\hat{j} + 6\hat{j} \times (-7.15\hat{i} - 6\hat{j})$$

In equation (1), the only term which contains a z-component is $\vec{\alpha} \times \vec{r}_{P/O}$, so we only need to consider this term.

Now,

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-\alpha_s \cos 40° \hat{i} + (-\alpha_s \sin 40° + 2)\hat{j} + 6\hat{j} \times (-7.15\hat{i} - 6\hat{j})\right) \times \frac{L}{\cos 40°}\hat{i}$$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-\alpha_s \cos 40^\circ \hat{i} + (-\alpha_s \sin 40^\circ + 2)\hat{j} + (6)(7.15)\hat{k} \right) \times \frac{L}{\cos 40^\circ}\hat{i}$$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left[-(-\alpha_s \sin 40^\circ + 2)\hat{k} + (6)(7.15)\hat{j} \right] \frac{L}{\cos 40^\circ}$$

The z-component term must be 0, so,

$$-\alpha_s \sin 40^\circ + 2 = 0$$

Solve for $\alpha_s = \dfrac{2}{\sin 40^\circ}$ rad/s^2

Therefore,

$$\vec{\alpha} = -\frac{2}{\sin 40^\circ}\cos 40^\circ \hat{i} + (6)(7.15)\hat{k}$$

$$\underline{\underline{\vec{\alpha} = -2.38\,\hat{i} + 42.9\,\hat{k}}} \text{ rad/s}^2 \text{ (answer)}$$

Note that L and r do not enter in the solution!