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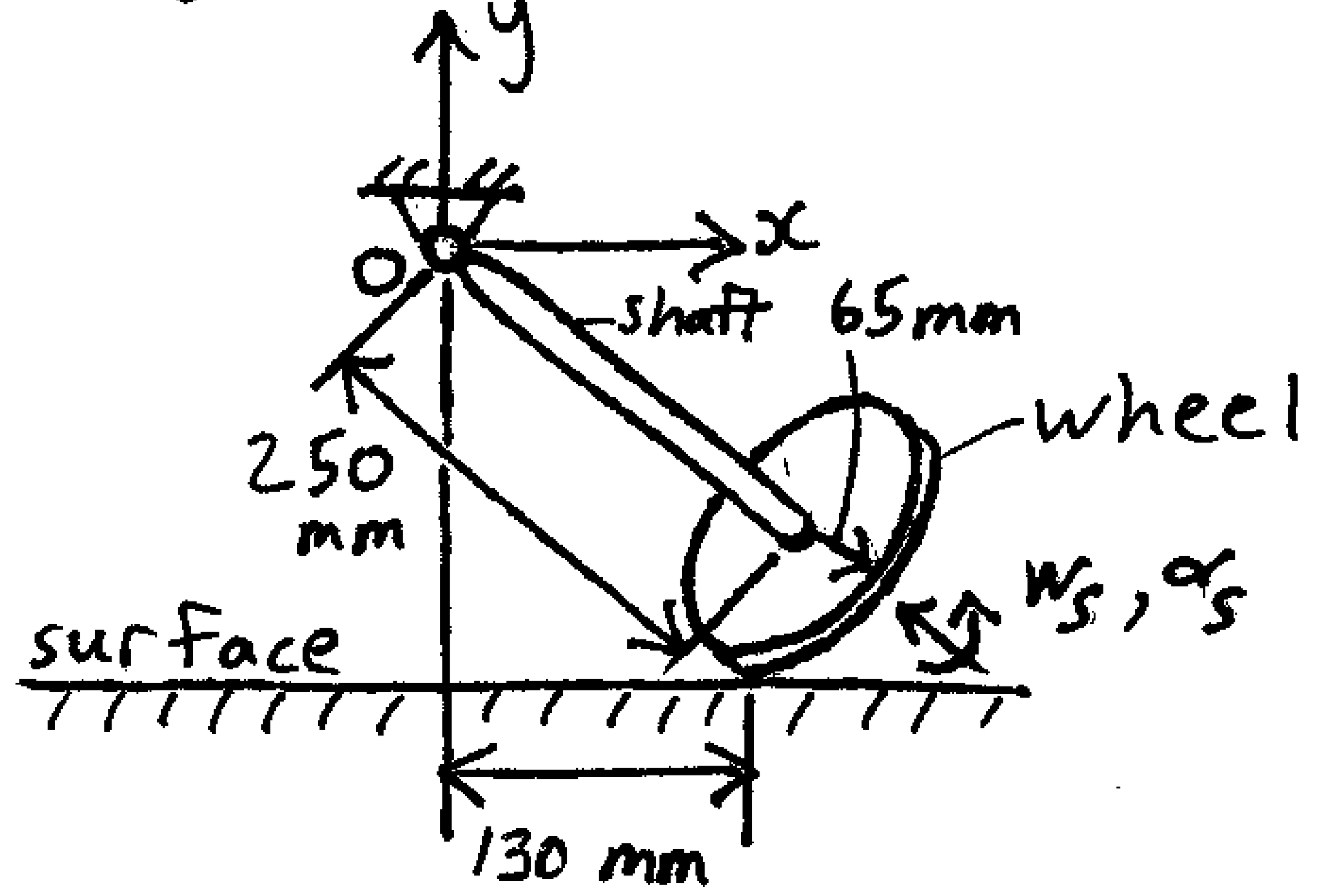
This is a 3D general motion problem (engineering mechanics).



A wheel rolls along a level surface without slipping. The wheel is connected to a shaft which has a ball-and-socket joint at O, around which the shaft and wheel rotate, with a spin angular velocity of $w_s = 7$ rad/s and a spin angular acceleration of $\alpha_s = 3$ rad/s^2. Determine the angular velocity and angular acceleration of the wheel, at the instant shown.

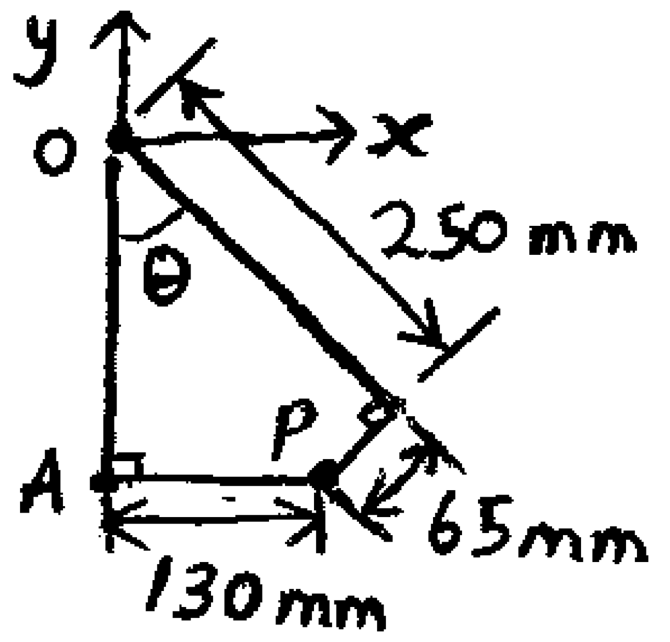
Solution:

Let P be the contact point between the wheel and the surface. Then,

$$\vec{V}_P = \underbrace{\vec{V}_O}_{0} + \vec{w} \times \vec{r}_{P/O} = 0$$, since the wheel rolls without slipping

Then, $\vec{w} \times \vec{r}_{P/O} = 0$

Geometry



$|OP| = \sqrt{65^2 + 250^2}$

$|OP| = 258.312\ mm$

$|OA| = \sqrt{|OP|^2 - |AP|^2}$

$|OA| = 223.215\ mm \qquad |AP| = 130\ mm$

$\vec{r}_{P/O} = 130\hat{i} - 223.215\hat{j}\ mm$

From before, $\vec{w} \times \vec{r}_{P/O} = \vec{w} \times (130\hat{i} - 223.215\hat{j}) = 0 \quad (1)$

Now, $\theta = \sin^{-1}\left(\frac{65}{258.312}\right) + \sin^{-1}\left(\frac{130}{258.312}\right)$

$\theta = 44.8°$

Next, $\vec{w} = -w_s \sin\theta\,\hat{i} + (w_s \cos\theta + w_y)\hat{j}$

→ where w_y is the precession rate of the wheel about O, unknown right now.

Substitute above into (1):

$$(-w_s \sin\theta\,\hat{i} + (w_s\cos\theta + w_y)\hat{j}) \times (130\hat{i} - 223.215\hat{j}) = 0$$

substitute known quantities:

$$(-4.932\hat{i} + (4.968 + w_y)\hat{j}) \times (130\hat{i} - 223.215\hat{j}) = 0$$

$$\Rightarrow (4.932)(223.215)\hat{k} - (4.968 + w_y)(130)\hat{k} = 0$$

Solve for $w_y = 3.5\ rad/s$

Therefore, $\underline{\underline{\vec{w} = -4.932\hat{i} + 8.468\hat{j}}}$ (answer) rad/s

Lastly, $\vec{a}_P = \underset{\substack{\downarrow \\ 0}}{\vec{a}_o} + \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$ (2)

The tangential component of $\vec{a}_P$ (along the z-direction) is 0, due to the no-slip condition.

Now, $\vec{\alpha} = -\alpha_s \sin\theta\,\hat{i} + (\alpha_s \cos\theta + \alpha_y)\,\hat{j}$

$+ \underbrace{\vec{w}_y \times \vec{w}_s}$ → where α_y is the precession angular acceleration of the wheel at point O, unknown right now.

$\vec{w}_s = -w_s \sin\theta\,\hat{i} + w_s \cos\theta\,\hat{j}$

$\vec{w}_y = 3.5\,\hat{j}$

The w_y rotation vector causes the rotation of the w_s vector. Note that the w_y rotation vector always points along y-axis (constant direction)

Substitute known quantities into above equation:

$$\vec{\alpha} = -3\sin 44.8°\,\hat{i} + (3\cos 44.8° + \alpha_y)\,\hat{j}$$
$$+ 3.5\,\hat{j} \times (-7\sin 44.8°\,\hat{i} + 7\cos 44.8°\,\hat{j})$$

In equation (2), the only term which contains a z-component is $\vec{\alpha} \times \vec{r}_{P/O}$, so we only need to consider this term.

Now,

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-3\sin 44.8^\circ \hat{i} + (3\cos 44.8^\circ + \alpha_y)\hat{j} + 3.5\hat{j} \times (-7\sin 44.8^\circ \hat{i} + 7\cos 44.8^\circ \hat{j})\right) \times (130\hat{i} - 223.215\hat{j})$$

$$\vec{\alpha} \times \vec{r}_{P/O} = \left(-3\sin 44.8^\circ \hat{i} + (3\cos 44.8^\circ + \alpha_y)\hat{j} + (3.5)(7)\sin 44.8^\circ \hat{k}\right) \times (130\hat{i} - 223.215\hat{j})$$

$$\vec{\alpha} \times \vec{r}_{P/O} = (3)(223.215)\sin 44.8^\circ \hat{k} - (3\cos 44.8^\circ + \alpha_y)(130)\hat{k} + (3.5)(7)(130)\sin 44.8\hat{j} + (3.5)(7)(223.215)\sin 44.8^\circ \hat{i}$$

The z-component term must be 0, so

$$(3)(223.215)\sin 44.8^\circ - (3\cos 44.8^\circ + \alpha_y)(130) = 0$$

Solve for $\alpha_y = 1.5 \text{ rad/s}^2$

Therefore, $\vec{\alpha} = -2.114\hat{i} + 3.629\hat{j} + 17.264\hat{k}$ (ans.) rad/s^2