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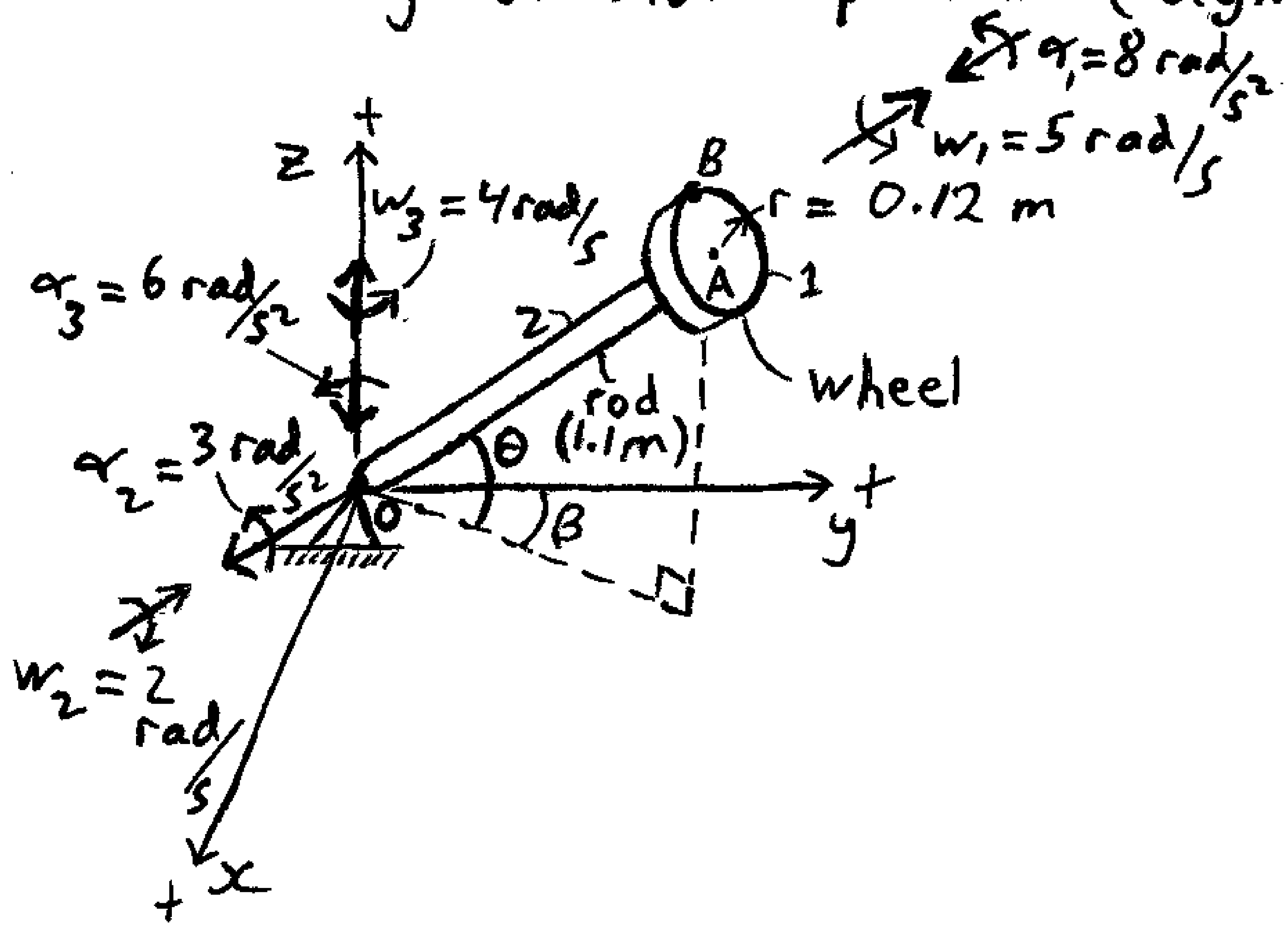
This is a 3D general motion problem (engineering mechanics).



A gyrotop is rotating as shown. The wheel on the end of the rod is being rotated with a motor which rotates with an angular velocity of $w_1 = 5\ rad/s$ and an angular acceleration of $\alpha_1 = 8\ rad/s^2$, in the rotational directions shown. At the same time, the rod attached to the wheel (OA) makes an angle $\theta = 65°$ with the xy-plane, and an angle $\beta = 35°$ with the y-axis, as shown. The angle θ decreases at the rate $w_2 = 2\ rad/s$, and the angular acceleration of θ is $3\ rad/s^2$, in the rotational directions shown. Lastly, the angle β decreases at the rate $w_3 = 4\ rad/s$, and the angular acceleration of β is $6\ rad/s^2$, in the rotational directions shown. If the radius of the wheel is 0.12 m, what is the velocity and acceleration of the topmost point of the wheel (point B)?

FYI

Note:
$w_2 = \dot{\theta}$, $w_3 = -\dot{\beta}$
$\alpha_2 = \ddot{\theta}$, $\alpha_3 = -\ddot{\beta}$

$w_1 \rightarrow$ spin
$w_2 \rightarrow$ nutation
$w_3 \rightarrow$ precession

Solution:

For the wheel, the angular velocity is the vector sum of all the contributing angular velocities. Therefore,

$$\vec{w}_w = (w_1 \cos\theta \sin\beta - w_2 \cos\beta)\,\hat{i} + (w_1 \cos\theta \cos\beta + w_2 \sin\beta)\,\hat{j} + (w_1 \sin\theta + w_3)\,\hat{k}$$

Substitute given quantities:

$$\vec{w}_w = (5\cos 65° \sin 35° - 2\cos 35°)\,\hat{i} + (5\cos 65° \cos 35° + 2\sin 35°)\,\hat{j} + (5\sin 65° + 4)\,\hat{k}$$

$$\vec{w}_w = -0.426\,\hat{i} + 2.878\,\hat{j} + 8.532\,\hat{k} \text{ rad/s} \quad (1)$$

For the rod, the angular velocity is the vector sum of all the contributing angular velocities (which does not include the angular velocity of the wheel). Therefore,

$$\vec{w}_r = (-w_2 \cos\beta)\,\hat{i} + (w_2 \sin\beta)\,\hat{j} + w_3\,\hat{k}$$

Substitute given quantities:

$$\vec{w}_r = -1.638\,\hat{i} + 1.147\,\hat{j} + 4\,\hat{k} \text{ rad/s} \quad (2)$$

For the wheel, the angular acceleration is as follows:

$$\vec{\alpha}_w = (-\alpha_1 \cos\theta \sin\beta + \alpha_2 \cos\beta)\,\hat{i} + (-\alpha_1 \cos\theta\cos\beta - \alpha_2 \sin\beta)\,\hat{j} + (-\alpha_1 \sin\theta - \alpha_3)\,\hat{k} + \underbrace{\vec{w}_3 \times \vec{w}_2}_{\text{w_3 vector rotates w_2 vector}} + \underbrace{\vec{w}_3 \times \vec{w}_1}_{\text{W_3 vector rotates w_1 vector}} + \underbrace{\vec{w}_2 \times \vec{w}_1}_{\text{w_2 vector rotates w_1 vector}} \qquad (A)$$

Now,

$$\vec{w}_1 = (w_1 \cos\theta \sin\beta)\,\hat{i} + (w_1 \cos\theta\cos\beta)\,\hat{j} + (w_1 \sin\theta)\,\hat{k}$$

$$\vec{w}_2 = (-w_2 \cos\beta)\,\hat{i} + (w_2 \sin\beta)\,\hat{j}$$

$$\vec{w}_3 = w_3\,\hat{k} \quad (\text{constant direction})$$

Substitute given quantities:

$$\vec{\alpha}_w = (-8\cos 65^\circ \sin 35^\circ + 3\cos 35^\circ)\,\hat{i} + (-8\cos 65^\circ \cos 35^\circ - 3\sin 35^\circ)\,\hat{j} + (-8\sin 65^\circ - 6)\,\hat{k}$$

$$+ 4\hat{k} \times [(-2\cos 35^\circ)\,\hat{i} + (2\sin 35^\circ)\,\hat{j}]$$

$$+ 4\hat{k} \times [(5\cos 65^\circ \sin 35^\circ)\,\hat{i} + (5\cos 65^\circ \cos 35^\circ)\,\hat{j} + (5\sin 65^\circ)\,\hat{k}]$$

$$+ [(-2\cos 35^\circ)\,\hat{i} + (2\sin 35^\circ)\,\hat{j}] \times [(5\cos 65^\circ \sin 35^\circ)\,\hat{i} + (5\cos 65^\circ \cos 35^\circ)\,\hat{j} + (5\sin 65^\circ)\,\hat{k}]$$

$\vec{\alpha}_w = -5.796\hat{i} + 1.229\hat{j} - 17.477\hat{k}$ (algebra not shown)
rad/s^2 (3)

For the rod, the angular acceleration is the vector sum of all the contributing angular accelerations (which does not include the angular velocity and angular acceleration of the wheel). We can simply set $w_1 = 0$ and $\alpha_1 = 0$ in equation (A). Therefore,

$\vec{\alpha}_r = -2.131\hat{i} - 8.274\hat{j} - 6\hat{k}$ rad/s^2 (algebra not shown)
(4)

Now, $\vec{V}_B = \vec{V}_A + \vec{w}_w \times \vec{r}_{B/A}$

and

$\vec{V}_A = \underbrace{\vec{V}_0}_{0} + \vec{w}_r \times \vec{r}_{A/0}$

and $\vec{r}_{A/0} = 1.1\cos\theta\sin\beta\,\hat{i} + 1.1\cos\theta\cos\beta\,\hat{j} + 1.1\sin\theta\,\hat{k}$

$\vec{r}_{B/A} = -0.12\sin\theta\sin\beta\,\hat{i} - 0.12\sin\theta\cos\beta\,\hat{j} + 0.12\cos\theta\,\hat{k}$

Therefore,

$\vec{V}_B = \vec{w}_r \times \vec{r}_{A/0} + \vec{w}_w \times \vec{r}_{B/A}$ (5)

Next,

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_w \times \vec{r}_{B/A} + \vec{\omega}_w \times (\vec{\omega}_w \times \vec{r}_{B/A})$$

and

$$\vec{a}_A = \underbrace{\vec{a}_o}_{0} + \vec{\alpha}_r \times \vec{r}_{A/O} + \vec{\omega}_r \times (\vec{\omega}_r \times \vec{r}_{A/O})$$

Therefore,

$$\vec{a}_B = \vec{\alpha}_r \times \vec{r}_{A/O} + \vec{\omega}_r \times (\vec{\omega}_r \times \vec{r}_{A/O}) + \vec{\alpha}_w \times \vec{r}_{B/A} + \vec{\omega}_w \times (\vec{\omega}_w \times \vec{r}_{B/A}) \qquad (6)$$

In equation (5), substitute given quantities to solve for velocity of point B:

$$\vec{V}_B = (-1.638\hat{i} + 1.147\hat{j} + 4\hat{k}) \times (1.1\cos 65^\circ \sin 35^\circ \hat{i} + 1.1\cos 65^\circ \cos 35^\circ \hat{j} + 1.1 \sin 65^\circ \hat{k})$$
$$+ (-0.426\hat{i} + 2.878\hat{j} + 8.532\hat{k}) \times (-0.12\sin 65^\circ \sin 35^\circ \hat{i} - 0.12 \sin 65^\circ \cos 35^\circ \hat{j} + 0.12\cos 65^\circ \hat{k})$$

$$\Rightarrow \vec{V}_B = 0.53\hat{i} + 2.19\hat{j} - 0.71\hat{k} \text{ m/s}$$ (algebra not shown)

(answer)

In equation (6), substitute given quantities to solve for acceleration of point B:

$$\vec{a}_B = (-2.131\hat{i} - 8.274\hat{j} - 6\hat{k}) \times (1.1\cos 65°\sin 35°\hat{i} + 1.1\cos 65°\cos 35°\hat{j} + 1.1\sin 65°\hat{k})$$

$$+ (-1.638\hat{i} + 1.147\hat{j} + 4\hat{k}) \times [(-1.638\hat{i} + 1.147\hat{j} + 4\hat{k}) \times (1.1\cos 65°\sin 35°\hat{i} + 1.1\cos 65°\cos 35°\hat{j} + 1.1\sin 65°\hat{k})]$$

$$+ (-5.796\hat{i} + 1.229\hat{j} - 17.477\hat{k}) \times (-0.12\sin 65°\sin 35°\hat{i} - 0.12\sin 65°\cos 35°\hat{j} + 0.12\cos 65°\hat{k})$$

$$+ (-0.426\hat{i} + 2.878\hat{j} + 8.532\hat{k}) \times [(-0.426\hat{i} + 2.878\hat{j} + 8.532\hat{k}) \times (-0.12\sin 65°\sin 35°\hat{i} - 0.12\sin 65°\cos 35°\hat{j} + 0.12\cos 65°\hat{k})]$$

$\Rightarrow$ $\vec{a}_B = -14.34\hat{i} + 6.69\hat{j} - 4.39\hat{k}$ m/s^2 (algebra not shown)

(answer)