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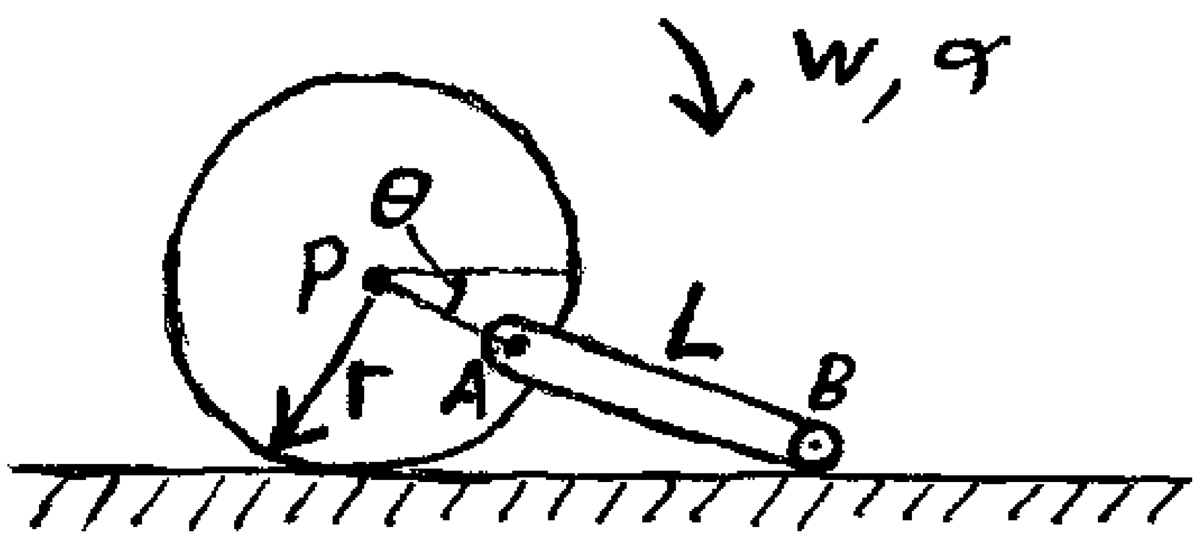
This is a 2D relative-motion problem involving acceleration (engineering mechanics).



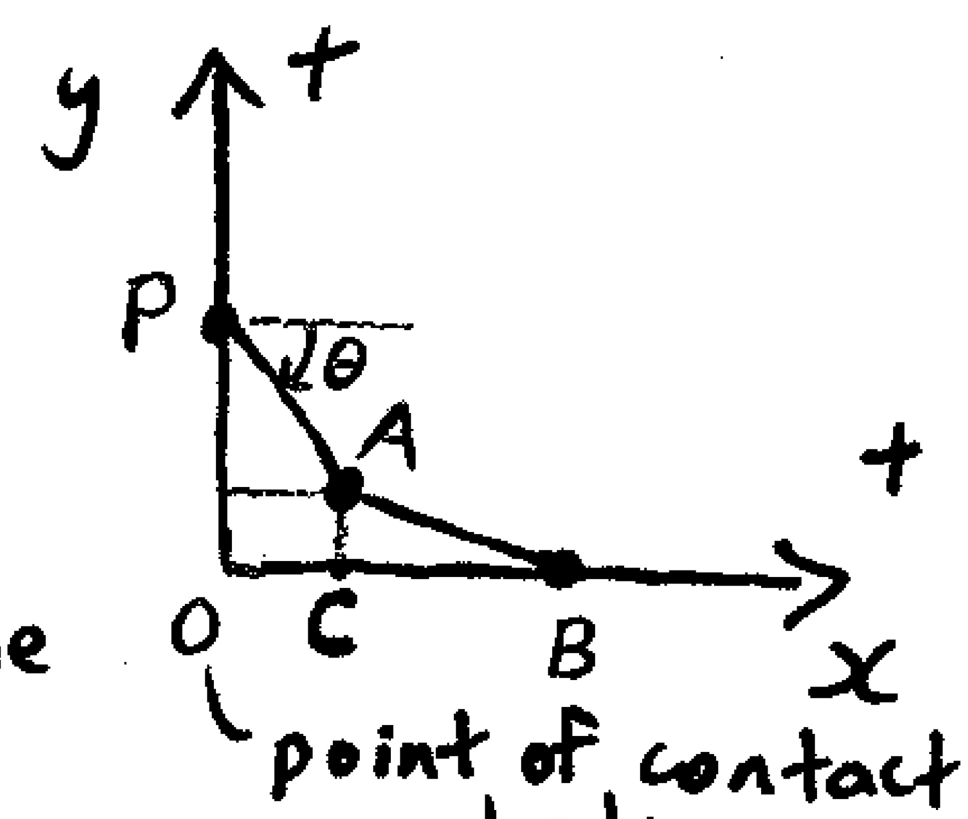
A wheel rolls without slipping on a flat surface with an angular velocity w, and an angular acceleration α. A rod is attached at point A and has a roller attached on the other end, at point B. The wheel has radius r and the rod has length L. What is the velocity and acceleration of point B when $\theta = 0°$?

Solution:

$\dot{\theta} \equiv w$

$\ddot{\theta} = \alpha$

θ varies with time



For this problem use a simple geometric approach rather than the usual vector equations.

$$OB = OC + CB$$

$$OC = AP\cos\theta = r\cos\theta \quad , \quad AC = OP - AP\sin\theta$$

$$AC = r - r\sin\theta$$

$$AC = r(1 - \sin\theta)$$

$$AB^2 = AC^2 + CB^2$$

$$L^2 = AC^2 + CB^2 \quad (I)$$

OB varies with time, so it can be differentiated with respect to time. The same goes for equation (I).

From equation (I):

$$\frac{d}{dt}(L^2) = \frac{d}{dt}(AC^2 + CB^2) \Rightarrow 0 = AC \cdot \frac{d(AC)}{dt} + CB \cdot \frac{d(CB)}{dt}$$

$$\frac{d(CB)}{dt} = \frac{-AC}{CB}\frac{d(AC)}{dt}$$

The velocity of point B is equal to $\frac{d(OB)}{dt} + v_p$.

Then,

$$\frac{d(OB)}{dt} = \frac{d(OC)}{dt} + \frac{d(CB)}{dt}$$

$$\frac{d(OB)}{dt} = -r\sin\theta \cdot \dot{\theta} - \frac{AC}{CB} \cdot \frac{d(AC)}{dt}$$

$$\frac{d(AC)}{dt} = -r\cos\theta \cdot \dot{\theta}$$

Then,

$$\frac{d(OB)}{dt} = -r\sin\theta \cdot \dot{\theta} + \frac{AC}{CB} \cdot r\cos\theta \cdot \dot{\theta} \quad (\mathrm{II})$$

when $\theta = 0°$, $AC = r$, $CB = \sqrt{L^2 - r^2}$

and, $$\frac{d(OB)}{dt} = \frac{r^2\dot{\theta}}{\sqrt{L^2 - r^2}}$$

(no slip condition) $v_p = \dot{\theta} r \rightarrow$ (velocity of point P)

Therefore, the velocity of point B is:

$$\Rightarrow \frac{d(OB)}{dt} + v_p = \frac{r^2\dot{\theta}}{\sqrt{L^2 - r^2}} + \dot{\theta} r \quad , \dot{\theta} \equiv \omega$$

(answer)

The acceleration of point B is equal to $\frac{d^2}{dt^2}(OB) + a_p$.

From equation (II):

$$\frac{d^2(OB)}{dt^2} = -r\left(\cos\theta\cdot(\dot{\theta})^2 + \sin\theta\cdot\ddot{\theta}\right) + \frac{d}{dt}\left(\frac{AC}{CB}\right)\cdot r\cos\theta\cdot\dot{\theta} + \frac{AC}{CB}\, r\left(-\sin\theta\cdot(\dot{\theta})^2 + \cos\theta\cdot\ddot{\theta}\right) \qquad (III)$$

Now,

$$\frac{d}{dt}\left(\frac{AC}{CB}\right) = \frac{d(AC)}{dt}\cdot\frac{1}{CB} - \frac{AC}{CB^2}\cdot\frac{d(CB)}{dt}$$

Substitute known quantities:

$$\frac{d}{dt}\left(\frac{AC}{CB}\right) = \frac{-r\cos\theta\cdot\dot{\theta}}{CB} - \frac{[r(1-\sin\theta)]^2\, r\cos\theta\cdot\dot{\theta}}{CB^3}$$

when $\theta = 0°$,

$$\frac{d}{dt}\left(\frac{AC}{CB}\right) = \frac{-r\dot{\theta}}{\sqrt{L^2-r^2}} - \frac{r^3\dot{\theta}}{\left(\sqrt{L^2-r^2}\right)^3}$$

From equation (III), when $\theta = 0°$, and sub. above equation:

$$\Rightarrow \frac{d^2(OB)}{dt^2} = -r(\dot{\theta})^2 - \frac{r^2(\dot{\theta})^2}{\sqrt{L^2-r^2}} - \frac{r^4(\dot{\theta})^2}{\left(\sqrt{L^2-r^2}\right)^3} + \frac{r^2\ddot{\theta}}{\sqrt{L^2-r^2}}$$

(no slip condition)

$a_p = \ddot{\theta} r \rightarrow$ (acceleration of point p)

The acceleration of point B is equal to

$\Rightarrow \dfrac{d^2(OB)}{dt^2} + a_p$, $\ddot{\theta} \equiv \alpha$, $\dot{\theta} \equiv w$ (answer).