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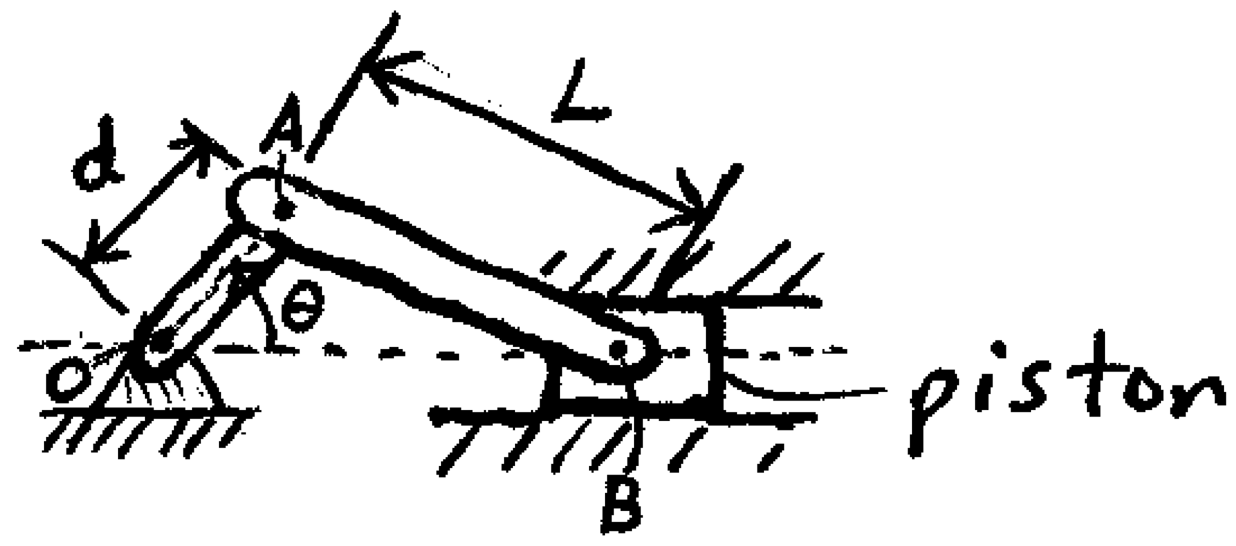
April 24, 2020

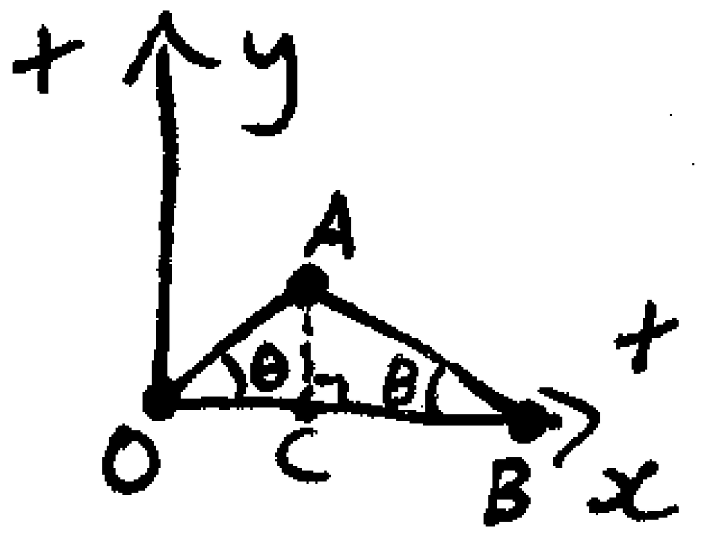
This is a 2D relative-motion problem involving acceleration (engineering mechanics).



A crank OA rotates counterclockwise at an angular velocity w and an angular acceleration α. Derive an expression for the velocity and acceleration of the piston.

<u>Solution</u>:

For this problem use a simple geometric approach rather than the usual vector equations.



θ varies with time
β varies with time
$\dot{\theta} \equiv w$
$\ddot{\theta} \equiv \alpha$

$$OB = OC + CB$$

$$OB = OA\cos\theta + AB\cos\beta$$

$$OB = d\cos\theta + L\cos\beta$$

OB varies with time, so it can be differentiated with respect to time.

The velocity of the piston is equal to $\dfrac{d}{dt}(OB)$.

$$\frac{d(OB)}{dt} = -d\sin\theta\cdot\dot{\theta} - L\sin\beta\cdot\dot{\beta} \quad (I)$$

From geometry,

$AC = OA\sin\theta = AB\sin\beta$

$\Rightarrow d\sin\theta = L\sin\beta \quad (II)$

Differentiate this with respect to time:

$d\cos\theta\cdot\dot{\theta} = L\cos\beta\cdot\dot{\beta} \quad (III)$

From (II): $\sin\beta = \frac{d\sin\theta}{L}$

From (III): $\dot{\beta} = \frac{d\cos\theta\cdot\dot{\theta}}{L\cos\beta}$

Substitute the above two equations into (I):

$$\frac{d(OB)}{dt} = -d\sin\theta\cdot\dot{\theta} - d\sin\theta\cdot\frac{d\cos\theta\cdot\dot{\theta}}{L\cos\beta}$$

$$\Rightarrow \frac{d(OB)}{dt} = -d\sin\theta\cdot\dot{\theta} - \frac{d^2\sin\theta\cos\theta\cdot\dot{\theta}}{L\cos\beta} \quad \text{(answer)}$$

(velocity of piston)

$(\dot{\theta} \equiv w)$

For a given value of θ, solve for β using equation (II)

The acceleration of the piston is equal to $\frac{d^2(OB)}{dt^2}$.

Differentiate equation (I) with respect to time:

$$\frac{d^2(OB)}{dt^2} = -d\left(\cos\theta\cdot(\dot{\theta})^2 + \sin\theta\cdot\ddot{\theta}\right) - L\left(\cos\beta\cdot(\dot{\beta})^2 + \sin\beta\cdot\ddot{\beta}\right) \quad (IV)$$

Differentiate equation (III) with respect to time:

$$d\left(-\sin\theta\cdot(\dot{\theta})^2 + \cos\theta\cdot\ddot{\theta}\right) = L\left(-\sin\beta\cdot(\dot{\beta})^2 + \cos\beta\cdot\ddot{\beta}\right) \quad (V)$$

From this,

$$\ddot{\beta} = \left[\frac{d}{L}\left(-\sin\theta\cdot(\dot{\theta})^2 + \cos\theta\cdot\ddot{\theta}\right) + \sin\beta\cdot(\dot{\beta})^2\right]\cdot\frac{1}{\cos\beta} \quad (VI)$$

For a given value of θ, solve for β using equation (II), then solve for $\dot{\beta}$ using equation (III), and then solve for $\ddot{\beta}$ using equation (VI).

Then solve for the acceleration of the piston using equation (IV), with $\dot{\theta} \equiv \omega$ and $\ddot{\theta} \equiv \alpha$. (answer)