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A particle travels along a path such that its acceleration is defined by $\vec{a} = 18\hat{i} + 2t\hat{j}$ ft/s². The particle starts from rest at $\vec{r} = 0$. When $t = 8$ s, calculate the speed of the particle and the radius of curvature of its path.

Solution:

$$a_x = 18 \text{ ft/s}^2 \quad (x\text{-component of acceleration})$$

$$a_y = 2t \text{ ft/s}^2 \quad (y\text{-component of acceleration})$$

Since $a_x = \dfrac{dv_x}{dt} = 18$

(x-component of velocity) $v_x = 18t + C_1$, C_1 is a constant of integration

Since $a_y = \dfrac{dv_y}{dt} = 2t$

(y-component of velocity) $v_y = t^2 + C_2$, C_2 is a constant of integration

Since $v_x = v_y = 0$ at $t = 0$, $C_1 = C_2 = 0$.

The speed of the particle is:

$$v = \sqrt{(v_x)^2 + (v_y)^2}$$

At $t = 8$ s, $v_x = 18(8) = 144$ ft/s

$v_y = (8)^2 = 64$ ft/s

$$v = \sqrt{(144)^2 + (64)^2} = 157.6 \text{ ft/s} \quad \text{(answer)}$$

The radius of curvature is given by the following formula:

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right| \qquad (1)$$

We don't have an expression for y in terms of x. So, use the chain rule of Calculus:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{V_y}{V_x} = \frac{t^2}{18t} = \frac{t}{18}$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{V_y}{V_x}\right)}{V_x}$$

$$= \frac{\frac{d}{dt}\left(t/18\right)}{18t}$$

$$= \frac{1}{18^2 t}$$

At $t = 8$ s,

$$\frac{dy}{dx} = \frac{8}{18} = 0.444$$

$$\frac{d^2y}{dx^2} = \frac{1}{18^2 \cdot 8} = 3.858 \times 10^{-4}$$

→ substitute into equation (1):

$\rho = 3395$ ft (answer)