In a game of pool, a cue ball strikes an object ball. Immediately after impact the cue ball has angular velocity $\vec{\omega}_0$ and center velocity $\vec{V}_0$, as shown. Given a coefficient of kinetic friction of $\mu_k$ between cue ball and table, calculate the trajectory of the cue ball after impact. Assume the pool table is perfectly flat.

Solution: We can further assume that the cue ball is a perfect sphere. This means that the principal moments of inertia are equal — that is, $I_x = I_y = I_z$.

Now, $\vec{V}_0$ has components $(V_{x0}, V_{y0})$, with $V_{z0} = 0$ with respect to ground.

$\vec{\omega}_0$ has components $(\omega_{x0}, \omega_{y0}, \omega_{z0})$.

The forces acting on the cue ball are gravity, the normal force ($N$) between ball and table, and the friction force ($F_s$) between ball and table. (As shown).
Immediately after impact there is relative motion (sliding) between ball and table. We can express this relative motion as \( \vec{V}_p \), where:

\[
\vec{V}_p = \vec{V} + \vec{w} \times \vec{CP} , \quad \vec{V}_p \neq 0
\]

where \( \vec{V} \) is the velocity of the ball center \( C \) at any time, and \( \vec{CP} \) is the vector from \( C \) to \( P \).

Now, \( \vec{CP} = -R \hat{k} \)

where \( R \) is the radius of the ball and \( \hat{k} \) is the unit vector pointing along the positive direction of \( Z \).

and

\[
\vec{V} = V_x \hat{i} + V_y \hat{j} \quad , \quad \vec{w} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}
\]

where \( \hat{i} \) is the unit vector pointing along the positive direction of \( X \), and \( \hat{j} \) is the unit vector pointing along the positive direction of \( Y \).

Therefore,

\[
\vec{V}_p = (V_x - R \omega_y) \hat{i} + (V_y + R \omega_x) \hat{j}
\]

Note that \( \omega_z \) does not show up, and does not affect the remainder of the analysis.
The magnitude of the kinetic friction is given as

$$|F_s| = \mu_k N$$  
(N is always pointing up magnitude) so we can drop the vector normal notation

Kinetic friction acts opposite the direction of (relative) motion, therefore

$$\vec{F}_s = - \frac{\vec{v}_p}{|\vec{v}_p|} \mu_k N$$

where

$$|\vec{v}_p| = \sqrt{(V_x - RW_y)^2 + (V_y + RW_x)^2}$$

The components along the X and Y directions are:

$$F_{sx} = - \frac{\mu_k N}{|\vec{v}_p|} (V_x - RW_y) \quad \text{(1)}$$

$$F_{sy} = - \frac{\mu_k N}{|\vec{v}_p|} (V_y + RW_x) \quad \text{(2)}$$

By Newton's second law,

$$F_{sx} = m \alpha_x,$$  \(\text{where } m \text{ is the mass of the ball, and } \alpha_x \text{ is the acceleration of mass center, along } X$$

$$F_{sy} = m \alpha_y,$$  \(\text{where } \alpha_y \text{ is the acceleration of mass center, along } Y$$

$$N - mg = m \alpha_z = 0,$$  \(\text{where } \alpha_z = 0 \text{ because ball does not move in } Z \text{- direction}$$

$$\therefore N = mg$$
We can apply Euler's equations of motion for a rigid body with $I_x = I_y = I_z = I$.

Therefore,

\[
\left(\sum \vec{M}_G\right)_x = I \dot{\theta}_x \quad \text{where the left side is the sum of the}
\]

\[
\left(\sum \vec{M}_G\right)_y = I \dot{\theta}_y
\]

\[
\left(\sum \vec{M}_G\right)_z = I \dot{\theta}_z \quad \text{moments about mass}
\]

\[
\text{center (G), and}
\]

\[
\dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z \quad \text{is the}
\]

\[
\text{angular acceleration of}
\]

\[
\text{the ball along } x, y, z \quad \text{respectively}
\]

\[
\sum \vec{M}_G = \vec{C}_p \times \left( F_{sx} \hat{x} + F_{sy} \hat{y} + N \hat{z} \right)
\]

from before $\vec{C}_p = -R \hat{z}$

Therefore,

\[
\sum \vec{M}_G = -RF_{sx} \hat{z} + RF_{sy} \hat{x}
\]

Thus,

\[
\left(\sum \vec{M}_G\right)_x = RF_{sy} \quad \left(\sum \vec{M}_G\right)_z = 0
\]

\[
\left(\sum \vec{M}_G\right)_y = -RF_{sx}
\]

Hence,

\[
RF_{sy} = I \dot{\theta}_x \quad (6)
\]

\[
-RF_{sx} = I \dot{\theta}_y \quad (7)
\]

\[
\dot{\theta}_z = 0
\]
Integrate equations (3) and (4) over time \( t \):
\[
\int_0^t F_{sx} \, dt = m \left( V_x - V_{x_0} \right) \quad (8)
\]
\[
\int_0^t F_{sy} \, dt = m \left( V_y - V_{y_0} \right) \quad (9)
\]

Similarly, integrate equations (6) and (7) over time \( t \):
\[
R \int_0^t F_{sy} \, dt = I \left( \dot{V}_x - \dot{V}_{x_0} \right) \quad (10)
\]
\[
-R \int_0^t F_{sx} \, dt = I \left( \dot{V}_y - \dot{V}_{y_0} \right) \quad (11)
\]

Substitute equations (8) and (9) into equations (10) and (11):
\[
R \, m \left( V_y - V_{y_0} \right) = I \left( \dot{V}_x - \dot{V}_{x_0} \right) \quad (12)
\]
\[-R \, m \left( V_x - V_{x_0} \right) = I \left( \dot{V}_y - \dot{V}_{y_0} \right) \quad (13)
\]

The ball stops sliding when \( \ddot{V}_\rho = 0 \). Thus,
\[
0 = \ddot{V} + \dot{\omega} \times \vec{c}_\rho
\]
\[
\ddot{V} = -\dot{\omega} \times \vec{c}_\rho = \dot{\omega} \times (R \vec{k}) \quad \text{for no sliding}
\]

\[
\dot{V}_x = R \, w_y \quad \text{for no sliding}
\]
\[
V_y = -R \, w_x
\]
Substitute $V_x$ and $V_y$ (for no sliding) into equations (12) and (13). We get

\[ Rm (V_y - V_y^o) = I \left( \frac{-V_x - Wx^o}{R} \right) \]

\[ -Rm (V_x - V_x^o) = I \left( \frac{V_y - Wy^o}{R} \right) \]

Solve for $V_x$ and $V_y$ in the above two equations. We get:

\[ V_x = \frac{mR V_x^o}{\frac{I}{R} + mR} + \frac{I Wy^o}{\frac{I}{R} + mR} \]

\[ V_y = \frac{mR V_y^o}{\frac{I}{R} + mR} - \frac{I Wx^o}{\frac{I}{R} + mR} \]

given $I = \frac{2}{5} m R^2$, we get

\[ V_x = \frac{5}{7} V_x^o + \frac{2}{7} R Wx^o \quad (14) \]

\[ V_y = \frac{5}{7} V_y^o - \frac{2}{7} R Wy^o \quad (15) \]

$V_x$ and $V_y$ define the final direction of the ball after impact. It is very interesting that this result does not depend on the frictional force $F_f$, and it does not depend on $\mu_k$. 
Now, let's consider the following elastic collision of a cue ball with an identical object ball, as shown.

Since we have conservation of momentum and kinetic energy,

\[ V_x = V \sin \phi \cos \phi, \quad V_y = V \sin^2 \phi \] (for the cue ball)

Immediately after impact, since the ball rolls without slipping before impact, \( w_x = -\frac{V}{R} \) (\( w_x, w_y, \) both equal to zero). Angular momentum does not change after impact.

Substitute \( V_x, V_y, w_x, w_y \) into equations (14) and (15). We get

\[ V_x = \frac{5}{7} V \sin \phi \cos \phi \]
\[ V_y = \frac{5}{7} V \sin^2 \phi + \frac{2}{7} V \]
Now,
\[
\theta_c = \tan^{-1}\left(\frac{V_x}{V_y}\right) = \tan^{-1}\left(\frac{5 \sin \phi \cos \phi}{5 \sin^2 \phi + 2}\right)
\]

The angle between the cue ball and the object ball is $\theta_c + \theta$. If there is no friction then the cue ball moves in a straight line after impact and the elastic collision equations can be used to solve for its trajectory after impact. Thus,

\[
\theta_c = \frac{\pi}{2} - \phi
\]

and $\theta_c + \phi = \frac{\pi}{2}$, as expected for an elastic collision.

Note that, for an initially stationary object ball, the angle $\phi$ is equal to the trajectory angle of the object ball, with or without friction.

From before,
\[
\vec{V}_p = (V_x - R\omega_y) \hat{i} + (V_y + R\omega_x) \hat{j}
\]

\[
V_{px} = V_x - R\omega_y
\]

\[
V_{py} = V_y + R\omega_x
\]
Take the derivative with respect to time.

\[ \dot{V}_{px} = \dot{V}_x - R \dot{W}_x \]
\[ \dot{V}_{py} = \dot{V}_y + R \dot{W}_x \]

Thus,

\[ \dot{V}_{px} = a_x - R \alpha_y \]
\[ \dot{V}_{py} = a_y + R \alpha_x \]

Substitute equations (3), (4) and (6), (7) into the above two equations. We get

\[ \dot{V}_{px} = \frac{F_{sx}}{m} + \frac{R^2 F_{sx}}{I} = F_{sx} \left( \frac{1}{m} + \frac{R^2}{I} \right) \]
\[ \dot{V}_{py} = \frac{F_{sy}}{m} + \frac{R^2 F_{sy}}{I} = F_{sy} \left( \frac{1}{m} + \frac{R^2}{I} \right) \]

with \( I = \frac{2}{5} m R^2 \)

\[ \dot{V}_{px} = F_{sx} \left( \frac{1}{m} + \frac{5}{2m} \right) = \frac{F_{sx}}{m} \cdot \frac{7}{2} \]
\[ \dot{V}_{py} = \frac{F_{sy}}{m} \cdot \frac{7}{2} \]

Substitute equations (1) and (2) into the above two equations (for \( N = mg \)). We get
\[
\dot{V}_{pX} = -\frac{\tau \mu_k g}{2|V_p|} \left( V_x - R w_y \right)
\]
\[
\dot{V}_{pY} = -\frac{\tau \mu_k g}{2|V_p|} \left( V_y + R w_x \right)
\]

Thus,
\[
\dot{V}_{pX} = \frac{\tau}{V_{pX}} V_{pX}
\]
\[
\dot{V}_{pY} = \frac{\tau}{V_{pY}} V_{pY}
\]
where \( \tau = \frac{\tau \mu_k g}{2|V_p|} \)

\[
\frac{\dot{V}_{pX}}{V_{pX}} = \frac{\dot{V}_{pY}}{V_{pY}}
\]

which becomes
\[
\frac{dV_{pX}}{dV_{pY}} = \frac{V_{pX}}{V_{pY}} \implies \frac{dV_{pX}}{V_{pX}} = \frac{dV_{pY}}{V_{pY}}
\]

Integrate:
\[
\ln(V_{pX}) = \ln(V_{pY}) + C_i
\]
Hence,

\[ V_{px} = V_{py} \cdot e^{C_1} \]

\[ V_{px} = V_{py} \cdot C_2 \], where \( C_2 = e^{C_1} \), constant

\[ \because \frac{V_{px}}{V_{py}} = C_2 \], constant

This tells us that the slip direction does not change, which means that the direction of the friction force \( F_s \), also does not change. Very interesting!

Since the friction force \( F_s \) is constant in magnitude and direction then the cue ball trajectory, after impact, will be parabolic (by analogy with projectile motion for constant downward \( g \)). The trajectory will be parabolic until sliding stops.

(Note that \( F_s \) is constant in magnitude because from before, \( |F_s| = \mu_k N \), with \( N = mg \) which is constant).

Since \( V_{px} \) is constant \[ \frac{V_{px}}{V_{py}} = \frac{V_{x_0} - RW_{y_0}}{V_{y_0} + RW_{x_0}} = C_2 \]

From equations (1) and (2),

\[ F_{sx} = -\mu_k N \cdot \frac{C_2}{\sqrt{1 + C_2^2}} \], \[ F_{sx} = -\mu_k mg \cdot \frac{C_2}{\sqrt{1 + C_2^2}} \]
$F_{sy} = -\mu_k mg \frac{1}{\sqrt{1+c_e^2}}$

Substitute $F_{sx}$, $F_{sy}$ into equations (3) and (4).
We get

$$-\mu_k mg \frac{c_e}{\sqrt{1+c_e^2}} = ma_x$$

$$-\mu_k mg \frac{c_e}{\sqrt{1+c_e^2}} = ma_y$$

Therefore,

$$a_x = -\mu_k g \frac{c_e}{\sqrt{1+c_e^2}} \quad , \quad a_y = -\mu_k g \frac{c_e}{\sqrt{1+c_e^2}} \quad \text{both constant}$$

Integrate:

$$V_x = -\mu_k g \frac{c_e}{\sqrt{1+c_e^2}} \cdot t + A_x$$

$$V_y = -\mu_k g \frac{c_e}{\sqrt{1+c_e^2}} \cdot t + A_y$$

During sliding stage

At $t=0$, $V_x = V_{x0}$, $V_y = V_{y0}$ (initial conditions)

Therefore,

$$V_x = -\mu_k g \frac{c_e \cdot t + V_{x0}}{\sqrt{1+c_e^2}} \quad V_y = -\mu_k g \frac{c_e \cdot t + V_{y0}}{\sqrt{1+c_e^2}}$$
To find the duration (time) of sliding, substitute the velocities $V_x$ and $V_y$ from equations (14) and (15) into the above two equations and solve for $t$. Of course, both these equations will give the same time $t$, which is

$$t = \frac{2 \sqrt{(V_{x_0} - Rw_{y_0})^2 + (V_{y_0} + Rw_{x_0})^2}}{7\mu kg}$$

Integrate $V_x$ and $V_y$ to obtain cue ball position during sliding stage.

$$x(t) = -\frac{\mu_k g t^2}{2\sqrt{1 + \zeta^2}} + V_{x_0} t + (B_1)$$

$$y(t) = -\frac{\mu_k g t^2}{2\sqrt{1 + \zeta^2}} + V_{y_0} t + (B_2)$$

At $t=0$, set $x(t) = y(t) = 0$, hence $B_1 = B_2 = 0$

Thus,

$$x(t) = -\frac{\mu_k g t^2}{2\sqrt{1 + \zeta^2}} + V_{x_0} t$$

$$y(t) = -\frac{\mu_k g t^2}{2\sqrt{1 + \zeta^2}} + V_{y_0} t$$

Cue ball position during sliding stage, after impact.