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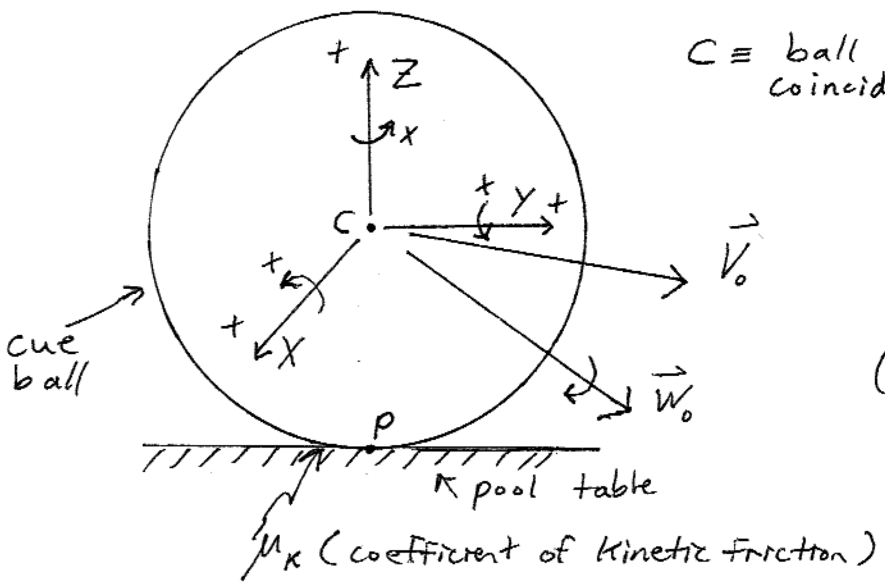
In a game of pool, a cue ball strikes an object ball. Immediately after impact the cue ball has angular velocity $\vec{w}_o$ and center velocity $\vec{V}_o$, as shown. Given a coefficient of kinetic friction of μ_k between cue ball and table, calculate the trajectory of the cue ball after impact.
Assume the pool table is perfectly flat.



C ≡ ball center, coincides with mass center (G)

↓ g (gravity)

XYZ is a (chosen) coordinate system with origin at C, with fixed orientation, and sign convention shown,

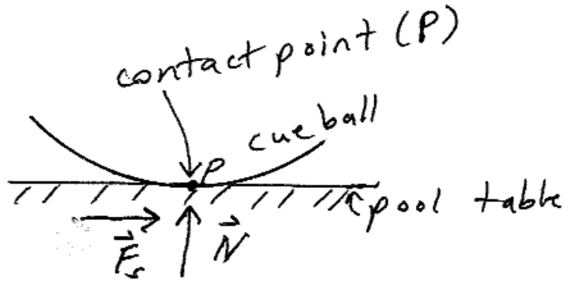
<u>Solution:</u> We can further assume that the cue ball is a perfect sphere. This means that the principal moments of inertia are equal - that is, $I_x = I_y = I_z$

Now, $\vec{V}_o$ has components (V_{xo}, V_{yo}), with $V_{zo} = 0$

$\vec{w}_o$ has components (w_{xo}, w_{yo}, w_{zo})

with respect to ground

The forces acting on the cue ball are gravity, the normal force (N) between ball and table, and the friction force (F_s) between ball and table (as shown).



$\vec{F}_s$ is the only force acting on the ball in the X Y plane

Immediately after impact there is relative motion (sliding) between ball and table. We can express this relative motion as $\vec{V}_p$, where:

$$\vec{V}_p = \vec{V} + \vec{w} \times \vec{CP} \quad , \quad \vec{V}_p \neq 0$$

where $\vec{V}$ is the velocity of the ball center C at any time, and $\vec{CP}$ is the vector from C to P.

Now, $\vec{CP} = -R\hat{K}$

where R is the radius of the ball and $\hat{K}$ is the unit vector pointing along the positive direction of Z.

and

$$\vec{V} = V_x \hat{I} + V_y \hat{J} \quad , \quad \vec{w} = \overbrace{w_x \hat{I} + w_y \hat{J} + w_z \hat{K}}^{\text{angular velocity of ball}}$$

where $\hat{I}$ is the unit vector pointing along the positive direction of X, and $\hat{J}$ is the unit vector pointing along the positive direction of Y.

Therefore,

$$\vec{V}_p = (V_x - Rw_y)\hat{I} + (V_y + Rw_x)\hat{J}$$

Note that w_z does not show up, and does not affect the remainder of the analysis

The magnitude of the kinetic friction is given as

$$|\vec{F}_s| = \mu_K N$$

N ↑ magnitude of normal force

(N is always pointing up so we can drop the vector notation)

Kinetic friction acts opposite the direction of (relative) motion, therefore

$$\vec{F}_s = \frac{-\vec{V}_P}{|\vec{V}_P|} \mu_K N$$

where

$$|\vec{V}_P| = \sqrt{(V_x - R\omega_y)^2 + (V_y + R\omega_x)^2}$$

The components along the X and Y directions are:

$$F_{sx} = \frac{-\mu_K N}{|\vec{V}_P|}(V_x - R\omega_y) \qquad (1)$$

$$F_{sy} = \frac{-\mu_K N}{|\vec{V}_P|}(V_y + R\omega_x) \qquad (2)$$

By Newton's second law,

(3) $F_{sx} = m a_x$, where m is the mass of the ball, and a_x is the acceleration of mass center, along X

(4) $F_{sy} = m a_y$, where a_y is the acceleration of mass center, along Y

(5) $N - mg = m a_z = 0$, where $a_z = 0$ because ball does not move in z-direction

$\therefore N = mg$

We can apply Euler's equations of motion for a rigid body with $I_x = I_y = I_z \equiv I$

Therefore,

$$(\Sigma \vec{M}_G)_x = I\alpha_x$$
$$(\Sigma \vec{M}_G)_y = I\alpha_y$$
$$(\Sigma \vec{M}_G)_z = I\alpha_z$$

where the left side is the sum of the moments about mass center (G), and $\alpha_x, \alpha_y, \alpha_z$ is the angular acceleration of the ball along X, Y, Z respectively

$$\Sigma \vec{M}_G = \vec{CP} \times (F_{sx}\hat{i} + F_{sy}\hat{j} + N\hat{k})$$

from before, $\vec{CP} = -R\hat{k}$

Therefore,

$$\Sigma \vec{M}_G = -RF_{sx}\hat{j} + RF_{sy}\hat{i}$$

Thus, $(\Sigma \vec{M}_G)_x = RF_{sy}$ $\quad (\Sigma \vec{M}_G)_z = 0$

$(\Sigma \vec{M}_G)_y = -RF_{sx}$

Hence,

$$RF_{sy} = I\alpha_x \quad (6)$$

$$-RF_{sx} = I\alpha_y \quad (7)$$

$$\alpha_z = 0$$

Integrate equations (3) and (4) over time t:

$$\int_0^t F_{sx}\,dt = m\left(V_x - V_{xo}\right) \quad (8)$$

$$\int_0^t F_{sy}\,dt = m\left(V_y - V_{yo}\right) \quad (9)$$

Similarly, integrate equations (6) and (7) over time t:

$$R\int_0^t F_{sy}\,dt = I\left(w_x - w_{xo}\right) \quad (10)$$

$$-R\int_0^t F_{sx}\,dt = I\left(w_y - w_{yo}\right) \quad (11)$$

Substitute equations (8) and (9) into equations (10) and (11):

$$Rm\left(V_y - V_{yo}\right) = I\left(w_x - w_{xo}\right) \quad (12)$$

$$-Rm\left(V_x - V_{xo}\right) = I\left(w_y - w_{yo}\right) \quad (13)$$

The ball stops sliding when $\vec{V}_p = 0$. Thus,

$$0 = \vec{V} + \vec{w} \times \vec{CP}$$

$$\vec{V} = -\vec{w} \times \vec{CP} = \vec{w} \times (R\hat{k}), \text{ for no sliding}$$

$$\therefore \left.\begin{aligned} V_x &= R w_y \\ V_y &= -R w_x \end{aligned}\right\} \text{for no sliding}$$

Substitute V_x and V_y (for no sliding) into equations (12) and (13). We get

$$Rm(V_y - V_{y0}) = I\left(-\frac{V_y}{R} - w_{x0}\right)$$

$$-Rm(V_x - V_{x0}) = I\left(\frac{V_x}{R} - w_{y0}\right)$$

Solve for V_x and V_y in the above two equations. We get:

$$V_x = \frac{mRV_{x0}}{\frac{I}{R} + mR} + \frac{I w_{y0}}{\frac{I}{R} + mR}$$

$$V_y = \frac{mRV_{y0}}{\frac{I}{R} + mR} - \frac{I w_{x0}}{\frac{I}{R} + mR}$$

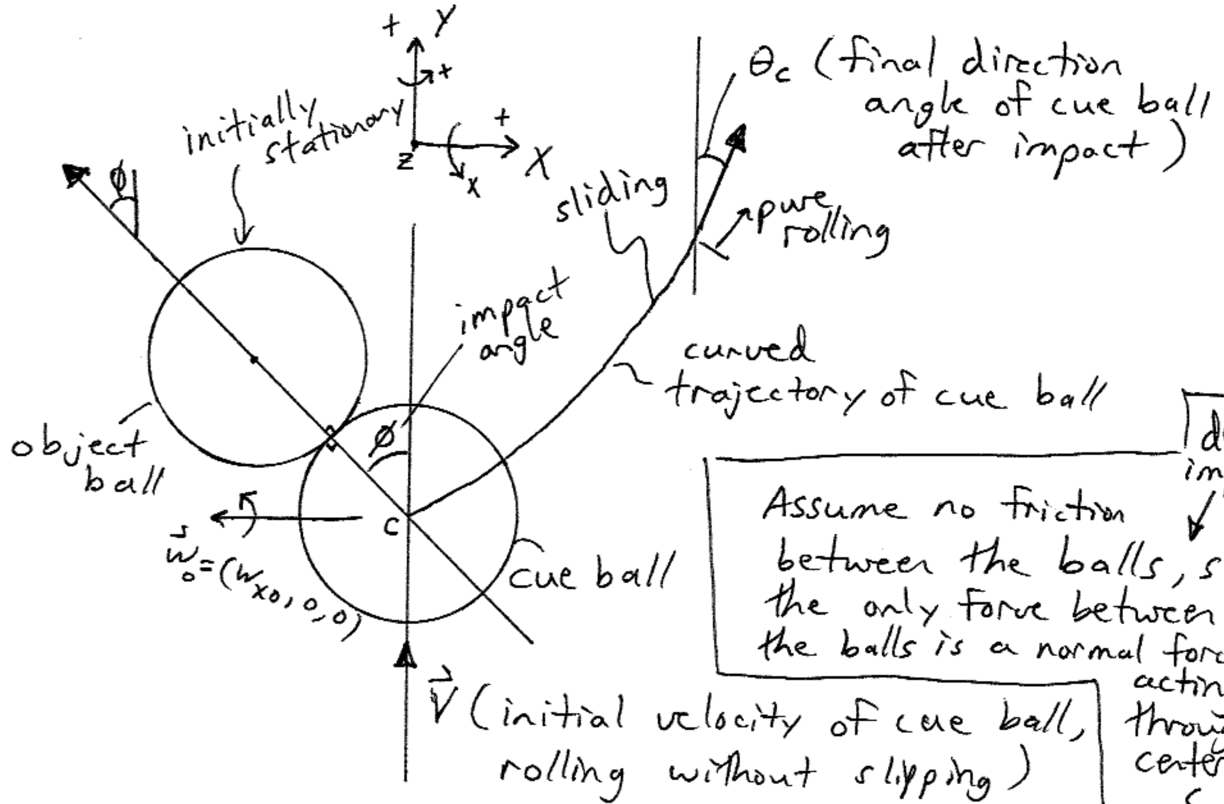
given $I = \frac{2}{5} m R^2$, we get

$$V_x = \frac{5}{7} V_{x0} + \frac{2}{7} R w_{y0} \quad (14)$$

$$V_y = \frac{5}{7} V_{y0} - \frac{2}{7} R w_{x0} \quad (15)$$

V_x and V_y define the final direction of the ball after impact. It is very intoesting that this result does not depend on the frictional force F_s, and it does not depend on μ_K.

Now, let's consider the following elastic collision of a cue ball with an identical object ball, as shown.



Assume no friction during impact between the balls, so the only force between the balls is a normal force acting through center c (G)

Thus, no moment exists to change angular momentum of balls during impact

Since we have conservation of momentum and kinetic energy,

immediately after impact

$$V_{x0} = V \sin\phi \cos\phi \quad , \quad V_{y0} = V \sin^2\phi \quad \text{(for the cue ball)}$$

V ↑ magnitude of $\vec{V}$ (initial velocity)

Since the ball rolls without slipping before impact,

$$\omega_{x0} = -\frac{V}{R} \quad , \quad (\omega_{y0}, \omega_{z0} \text{ both equal zero})$$

∴ Angular momentum does not change after impact.

Substitute V_{x0}, V_{y0}, ω_{x0}, $\omega_{y0}{=}0$ into equations (14) and (15). We get

$$V_x = \frac{5}{7} V \sin\phi \cos\phi$$

$$V_y = \frac{5}{7} V \sin^2\phi + \frac{2}{7} V$$

Now,

$$\theta_c = \tan^{-1}\left(\frac{V_x}{V_y}\right) = \tan^{-1}\left(\frac{5\sin\phi\cos\phi}{5\sin^2\phi + 2}\right)$$

The angle between the cue ball and the object ball is $\theta_c + \phi$. If there is no friction then the cue ball moves in a straight line after impact and the elastic collision equations can be used to solve for its trajectory after impact. Thus,

$$\theta_c = \frac{\pi}{2} - \phi$$

and $\theta_c + \phi = \frac{\pi}{2}$, as expected for an elastic collision.

Note that, for an initially stationary object ball, the angle ϕ is equal to the trajectory angle of the object ball, with or without table friction.

From before,

$$\vec{V}_p = (V_x - R w_y)\hat{I} + (V_y + R w_x)\hat{J}$$

$$V_{px} = V_x - R w_y$$

$$V_{py} = V_y + R w_x$$

Take the derivative with respect to time.

$$\dot{V}_{px} = \dot{V}_x - R\dot{w}_y$$

$$\dot{V}_{py} = \dot{V}_y + R\dot{w}_x$$

Thus,

$$\dot{V}_{px} = a_x - R\alpha_y$$

$$\dot{V}_{py} = a_y + R\alpha_x$$

Substitute equations (3),(4) and (6),(7) into the above two equations. We get

$$\dot{V}_{px} = \frac{F_{sx}}{m} + \frac{R^2 F_{sx}}{I} = F_{sx}\left(\frac{1}{m} + \frac{R^2}{I}\right)$$

$$\dot{V}_{py} = \frac{F_{sy}}{m} + \frac{R^2 F_{sy}}{I} = F_{sy}\left(\frac{1}{m} + \frac{R^2}{I}\right)$$

with $I = \frac{2}{5} mR^2$

$$\dot{V}_{px} = F_{sx}\left(\frac{1}{m} + \frac{5}{2m}\right) = \frac{F_{sx}}{m} \cdot \frac{7}{2}$$

$$\dot{V}_{py} = \frac{F_{sy}}{m} \cdot \frac{7}{2}$$

Substitute equations (1) and (2) into the above two equations (for $N = mg$). We get

$$\dot{V}_{px} = \frac{-7\mu_k g}{2|\vec{V}_p|}\overbrace{(V_x - R\omega_y)}^{V_{px}}$$

$$\dot{V}_{py} = \frac{-7\mu_k g}{2|\vec{V}_p|}\underbrace{(V_y + R\omega_x)}_{V_{py}}$$

Thus,

$$\dot{V}_{px} = ¢\, V_{px}$$

$$\dot{V}_{py} = ¢\, V_{py}$$

where $¢ = \dfrac{-7\mu_k g}{2|\vec{V}_p|}$

$$\frac{\dot{V}_{px}}{\dot{V}_{py}} = \frac{V_{px}}{V_{py}}$$

which becomes

$$\frac{dV_{px}}{dV_{py}} = \frac{V_{px}}{V_{py}} \implies \frac{dV_{px}}{V_{px}} = \frac{dV_{py}}{V_{py}}$$

Integrate:

$$\ln(V_{px}) = \ln(V_{py}) + \overset{\text{constant}}{C_1}$$

Hence,

$$V_{px} = V_{py} \cdot e^{C_1}$$

$$V_{px} = V_{py} \cdot C_2 \quad , \text{ where } C_2 = e^{C_1}, \text{ constant}$$

$$\therefore \quad \frac{V_{px}}{V_{py}} = C_2, \text{ constant}$$

This tells us that the slip direction does not change, which means that the direction of the friction force $\vec{F}_s$, also does not change. Very interesting!

Since the friction force $\vec{F}_s$ is constant in magnitude and direction then the cue ball trajectory, after impact, will be parabolic (by analogy, with projectile motion for constant downward g). The trajectory will be parabolic until sliding stops.

(Note that $\vec{F}_s$ is constant in magnitude because, from before, $|\vec{F}_s| = \mu_K N$, with $N = mg$ which is constant).

Since $\frac{V_{px}}{V_{py}}$ is constant $\frac{V_{px}}{V_{py}} = \frac{V_{x0} - R\,w_{y0}}{V_{y0} + R\,w_{x0}} = C_2$

From equations (1) and (2),

$$F_{sx} = -\mu_K N \cdot \frac{C_2}{\sqrt{1+C_2^2}} \quad , \quad F_{sx} = -\mu_K mg \frac{C_2}{\sqrt{1+C_2^2}}$$

$$F_{sy} = -\mu_K mg \cdot \frac{1}{\sqrt{1+C_2^2}}$$

Substitute F_{sx}, F_{sy} into equations (3) and (4). We get

$$\frac{-\mu_K mg\, C_2}{\sqrt{1+C_2^2}} = m\, a_x$$

$$\frac{-\mu_K mg}{\sqrt{1+C_2^2}} = m\, a_y$$

Therefore,

$$a_x = \frac{-\mu_K g\, C_2}{\sqrt{1+C_2^2}}, \quad a_y = \frac{-\mu_K g}{\sqrt{1+C_2^2}} \Bigg\} \text{ both constant}$$

Integrate;

$$V_x = \frac{-\mu_K g\, C_2}{\sqrt{1+C_2^2}} \cdot t + A_1$$

$$V_y = \frac{-\mu_K g}{\sqrt{1+C_2^2}} \cdot t + A_2$$

constants

During sliding stage

At $t=0$, $V_x = V_{xo}$, $V_y = V_{yo}$ (initial conditions)

Therefore,

$$V_x = \frac{-\mu_K g\, C_2}{\sqrt{1+C_2^2}} \cdot t + V_{xo}, \quad V_y = \frac{-\mu_K g}{\sqrt{1+C_2^2}} \cdot t + V_{yo}$$

To find the duration (time) of sliding substitute the velocities V_x and V_y from equations (14) and (15) into the above two equations and solve for t. Of course, both these equations will give the same time t, which is

$$t = \frac{2\sqrt{(V_{x0} - R\omega_{y0})^2 + (V_{y0} + R\omega_{x0})^2}}{7\mu_K g}$$

Integrate V_x and V_y to obtain cue ball position during sliding stage.

$$x(t) = \frac{-\mu_K g\, C_2 \cdot t^2}{2\sqrt{1+C_2^2}} + V_{x0}\, t + B_1$$

constants

$$y(t) = \frac{-\mu_K g \cdot t^2}{2\sqrt{1+C_2^2}} + V_{y0}\, t + B_2$$

At $t=0$ set $x(t) = y(t) = 0$, hence $B_1 = B_2 = 0$

Thus,

$$\left.\begin{aligned} x(t) &= \frac{-\mu_K g\, C_2 \cdot t^2}{2\sqrt{1+C_2^2}} + V_{x0}\, t \\ y(t) &= \frac{-\mu_K g \cdot t^2}{2\sqrt{1+C_2^2}} + V_{y0}\, t \end{aligned}\right\}$$

Cue ball position during sliding stage, after impact