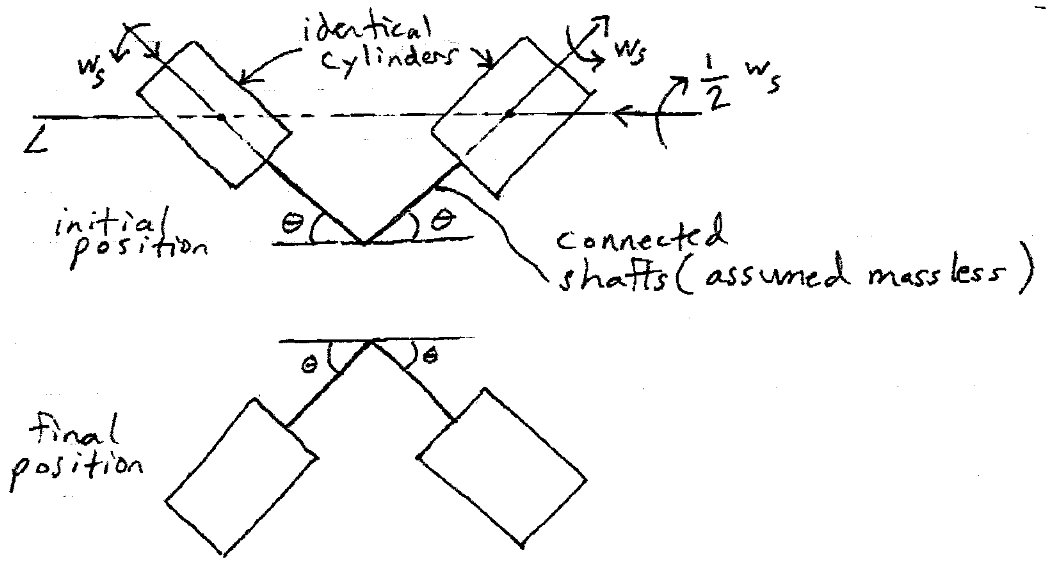
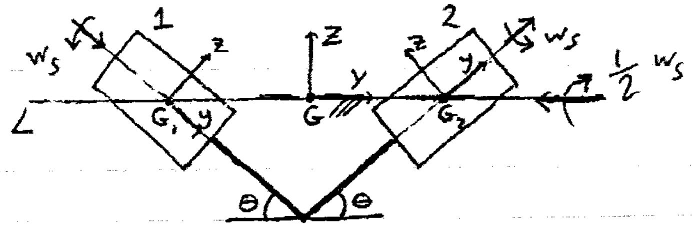
The cat righting reflex allows a cat to flip 180° from an upside down position, upon falling, so that it can land on its feet. By twisting a certain way, the cat can do this while maintaining zero angular momentum. To help understand how this is possible consider the following physical system shown below, representing the cat.



Relative to the shafts the cylinders spin at an angular velocity w_s, while the shafts spin in the opposite sense with angular velocity $\frac{1}{2} w_s$, with respect to ground. Find angle θ so that this is possible. The mass of the cylinders is m, the radius of the cylinders is a, and their length is $3a$.

Solution: Set up the analysis as follows.



G_1 is the center of mass of cylinder 1

G_2 is the center of mass of cylinder 2

G is the center of mass of the entire system

Place local yz frames at the center of mass of the two cylinders, as shown.
Place global (ground) frame (YZ) at center of mass of entire system, as shown.

No external moments (torque) acts on the system, therefore angular momentum is conserved for the cylinder-shaft system (representing the cat). Thus,

$$\underbrace{\sum \vec{M}_G = \frac{d\vec{H}_G}{dt}}_{\text{with respect to ground (an inertial reference frame)}} = 0$$
for the system of particles (in the cylinder-shaft system)

Therefore, $\vec{H}_G = \text{constant} = 0$ (zero angular momentum)

where $\sum \vec{M}_G$ is the sum of the moments, and $\vec{H}_G$ is the angular momentum of the system, about G.

The symmetry of the system allows it to rotate to the final position, while G_1, G_2, G all remain on line L.

The angular momentum of the entire system is given by,

$$\underbrace{\vec{H}_G}_{\text{relative to global YZ frame}} = \vec{H}_{G_1} + \vec{H}_{G_2} + \vec{r}_1 \times m_1 \overset{\nearrow 0}{\vec{V}_{G_1}} + \vec{r}_2 \times m_2 \overset{\nearrow 0}{\vec{V}_{G_2}}$$

where $\vec{H}_{G_1}$ is the angular momentum of cylinder 1, with respect to G_1
$\vec{H}_{G_2}$ is the angular momentum of cylinder 2, with respect to G_2
$\vec{r}_1, \vec{r}_2$ are the position vectors measured from origin of YZ to center of mass G_1 and G_2, respectively
m_1, m_2 are the masses of cylinders 1 and 2, respectively, $m_1 = m_2 = m$
$\vec{V}_{G_1}, \vec{V}_{G_2}$ are the velocities of cylinders 1 and 2, respectively, with respect to ground, at G_1, G_2
$\vec{V}_{G_1} = \vec{V}_{G_2} = 0$

Hence,

$$\vec{H}_G = \vec{H}_{G_1} + \vec{H}_{G_2} = 0 \quad (1)$$

The angular momentum of cylinder 1, with respect to G_1, and relative to its local yz frame, is given by:

$$\vec{H}_{G_1(yz)} = I_{y_1} w_{y_1} \hat{j}_1 + I_{z_1} w_{z_1} \hat{k}_1$$

where $I_{y_1} = \frac{1}{2} m a^2$

$$I_{z_1} = \frac{1}{12} m (3a^2 + 9a^2) = ma^2$$

with respect to ground, along local yz frame

$$\begin{cases} w_{y_1} = w_s - \frac{1}{2} w_s \cos\theta \\ w_{z_1} = -\frac{1}{2} w_s \sin\theta \end{cases}$$

$\hat{j}_1$ is the unit vector pointing along positive (local y for cylinder) $\hat{k}_1$ is the unit vector pointing along positive local z for cylinder 1

Similarly,

$$\vec{H}_{G_2(yz)} = I_{y_2} w_{y_2} \hat{j}_2 + I_{z_2} w_{z_2} \hat{k}_2$$

where I_{y_2} and I_{z_2} are the same as before, and

with respect to ground, along local yz frame

$$\begin{cases} w_{y_2} = w_s - \frac{1}{2} w_s \cos\theta \\ w_{z_2} = \frac{1}{2} w_s \sin\theta \end{cases}$$

Now, we must resolve $\vec{H}_{G_1(yz)}$ onto the YZ frame to give $\vec{H}_{G_1}$ which can be used in equation (1).

$$\vec{H}_{G_1} = \left(I_{y_1} w_{y_1} \cos\theta + I_{z_1} w_{z_1} \sin\theta \right) \hat{J} + \left(-I_{y_1} w_{y_1} \sin\theta + I_{z_1} w_{z_1} \cos\theta \right) \hat{K}$$

where $\hat{J}$ is the unit vector pointing along positive Y, and $\hat{K}$ is the unit vector pointing along positive Z

Similarly,

$$\vec{H}_{G_2} = \left(I_{y_2} w_{y_2} \cos\theta - I_{z_2} w_{z_2} \sin\theta \right) \hat{J} + \left(I_{y_2} w_{y_2} \sin\theta + I_{z_2} w_{z_2} \cos\theta \right) \hat{K}$$

From equation (1),

$$\vec{H}_G = \vec{H}_{G_1} + \vec{H}_{G_2}$$

$$= \left[I_{y_1} \left(w_s - \tfrac{1}{2} w_s \cos\theta \right) \cos\theta - I_{z_1} \cdot \tfrac{1}{2} w_s \sin\theta \cdot \sin\theta \right] \hat{J}$$
$$+ \left[-I_{y_1} \left(w_s - \tfrac{1}{2} w_s \cos\theta \right) \sin\theta - I_{z_1} \cdot \tfrac{1}{2} w_s \sin\theta \cdot \cos\theta \right] \hat{K}$$
$$+ \left[I_{y_2} \left(w_s - \tfrac{1}{2} w_s \cos\theta \right) \cos\theta - I_{z_2} \cdot \tfrac{1}{2} w_s \sin\theta \cdot \sin\theta \right] \hat{J}$$
$$+ \left[I_{y_2} \left(w_s - \tfrac{1}{2} w_s \cos\theta \right) \sin\theta + I_{z_2} \cdot \tfrac{1}{2} w_s \sin\theta \cdot \cos\theta \right] \hat{K}$$
$$= 0 \qquad (2)$$

Substitute $I_{y_1} = I_{y_2} = \frac{1}{2} m a^2$

and $I_{z_1} = I_{z_2} = m a^2$

and solve for θ in equation (2)

Solving, we get $\underline{\theta = 42.9^\circ}$

This problem was taken from "Vector Mechanics for Engineers", dynamics, second edition, Beer and Johnston, problem 18.28, pg 904.