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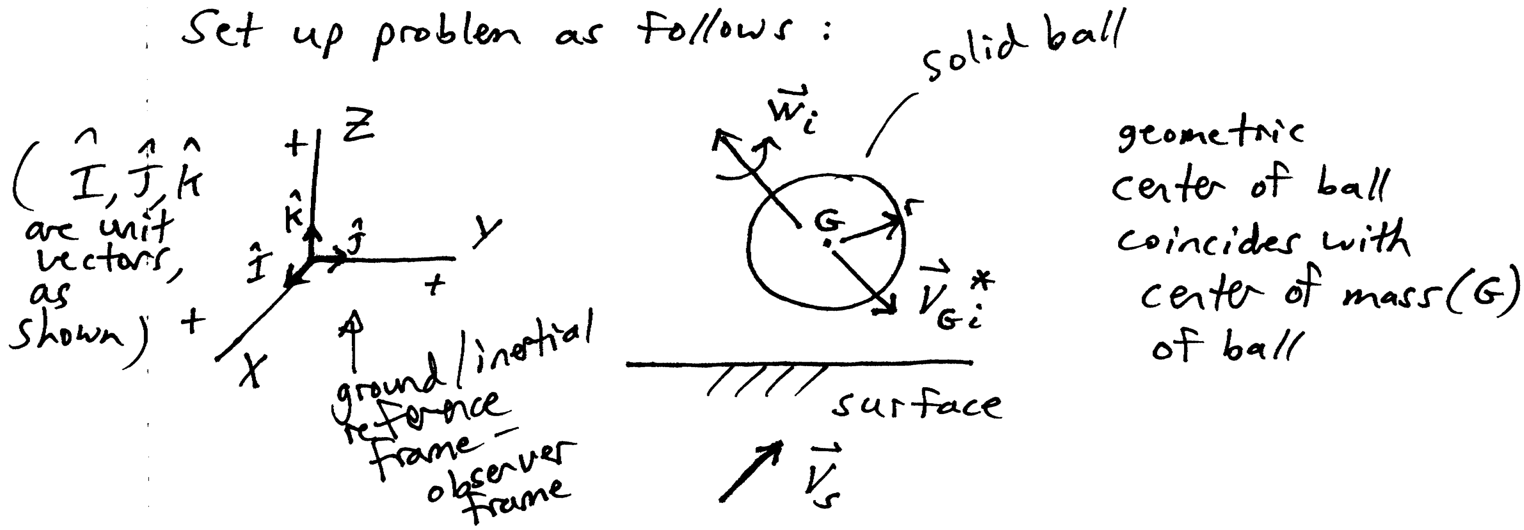
Spherical ball colliding with moving surface

Set up problem as follows:



where:

- The surface is in XY plane
- $\vec{w}_i$ is the angular velocity of the ball just before it strikes the surface
- $\vec{V}^*_{Gi}$ is the velocity of ball (at point G) just before it strikes the surface
- $\vec{V}_s$ is the velocity of the surface before the ball strikes, at the point of impact
- r is the sphere radius

Assumptions:

- The moving surface remains at a constant velocity during the brief impact duration. The surface is rigid
- Gravity has a negligible effect during the brief impact duration, in which impulse forces dominate
- The spherical ball can be treated as a rigid body with negligible deformation during impact

Now, $\vec{V}^*_{Gi} = V^*_{GXi}\hat{I} + V^*_{GYi}\hat{J} + V^*_{GZi}\hat{k}$

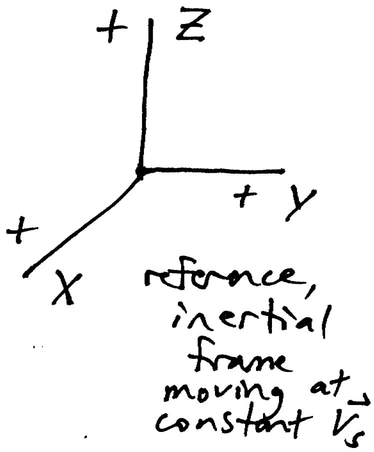
$$\vec{w}_i = w_{X_i}\hat{I} + w_{Y_i}\hat{J} + w_{Z_i}\hat{K}$$

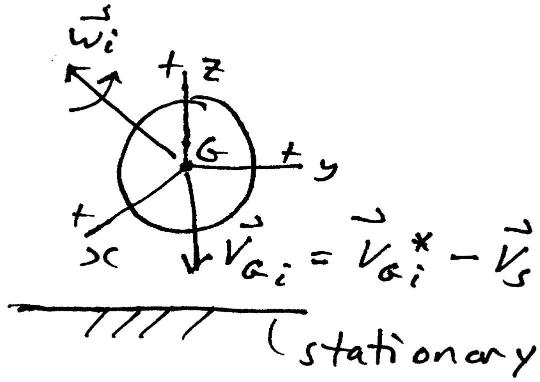
$$\vec{V}_s = V_{sX}\hat{I} + V_{sY}\hat{J} + V_{sZ}\hat{K}$$

$\vec{w}_i$, $\vec{V}^*_{G_i}$, $\vec{V}_s$ are with respect to ground, which is the reference frame we, as observers, are in.

For convenience, set the reference frame so that it moves at constant velocity $\vec{V}_s$. Then, relative to this reference frame the surface is stationary.

This problem is then transformed into a simpler one:





xyz is parallel to XYZ

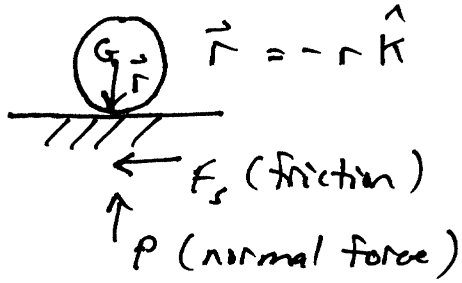
Where:

∴ xyz components are the same as XYZ components

- The local xyz frame has origin at center of mass of ball (G). It is assigned zero rotation which allows for easier solution since the sphere has equal principal moments of inertia about G for any orientation of xyz
- $\vec{V}_{G_i}$ is the velocity of ball (at point G) just before it strikes the surface. This velocity is relative to the reference frame moving at velocity $\vec{V}_s$

Now, $\vec{V}_{G_i} = V_{GX_i}\hat{I} + V_{GY_i}\hat{J} + V_{GZ_i}\hat{K}$

Break the problem into two stages from initial, just before impact, to point at which $V_{Gz} = 0$.
Apply impulse and momentum equations; given below.



$$\vec{r} = -r\hat{k}$$

$$\vec{M}_G = -r\hat{k} \times (P\hat{k} + F_{sx}\hat{I} + F_{sy}\hat{J})$$

(moment about G)

where F_s is the friction force along the surface, with components $F_{sx}\hat{I} + F_{sy}\hat{J}$, and P is the normal force along Z-direction.

Now, $\vec{M}_G = \underbrace{-r F_{sx}}_{M_{Gy}} \hat{J} + \underbrace{r F_{sy}}_{M_{Gx}} \hat{I}$ $\qquad M_{Gz} = 0$

$$\int M_{Gx}\, dt = I w_{x_2} - I w_{x_1} \quad (1)$$

$$\int M_{Gy}\, dt = I w_{y_2} - I w_{y_1} \quad (2)$$

$$\int M_{Gz}\, dt = I w_{z_2} - I w_{z_1} \quad (3)$$

$$\int F_{sx}\, dt = m V_{Gx_2} - m V_{Gx_1} \quad (4)$$

$$\int F_{sy}\, dt = m V_{Gy_2} - m V_{Gy_1} \quad (5)$$

$$\int P\, dt = m \underbrace{(0)}_{V_{Gz_2}} - m V_{Gz_1} \quad (6)$$

where I is the principal moment of inertia of the ball about G, and m is the ball mass

Hence,

$$(1) \Rightarrow r\int F_{sy}\,dt = I\omega_{x_2} - I\omega_{x_1}$$

$$(2) \Rightarrow -r\int F_{sx}\,dt = I\omega_{y_2} - I\omega_{y_1}$$

$$(3) \Rightarrow 0 = I\omega_{z_2} - I\omega_{z_1} \rightarrow \omega_{z_1} = \omega_{z_2}$$

$$(4) \Rightarrow \int F_{sx}\,dt = mV_{Gx_2} - mV_{Gx_1}$$

$$(5) \Rightarrow \int F_{sy}\,dt = mV_{Gy_2} - mV_{Gy_1}$$

$$(6) \Rightarrow \int P\,dt = -mV_{Gz_1}$$

initial stage of impact

The second (final) stage of impact is from the point at which $V_{Gz} = 0$ to the point immediately after the ball has left the surface. The above equations for this stage become:

$$(1) \Rightarrow r\int F_{sy}\,dt = I\omega_{x_3} - I\omega_{x_2}$$

$$(2) \Rightarrow -r\int F_{sx}\,dt = I\omega_{y_3} - I\omega_{y_2}$$

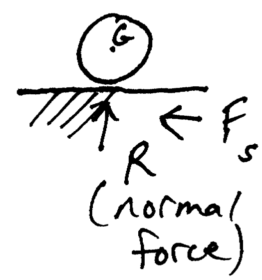
$$(3) \Rightarrow \omega_{z_2} = \omega_{z_3}$$

$$(4) \Rightarrow \int F_{sx}\,dt = mV_{Gx_3} - mV_{Gx_2}$$

$$(5) \Rightarrow \int F_{sy}\,dt = mV_{Gy_3} - mV_{Gy_2}$$

$$(6) \Rightarrow \int R\,dt = mV_{Gz_3} - \underbrace{0}_{V_{Gz_2}} = mV_{Gz_3}$$

Final stage of impact



← replace P in the above equation (6) with R

We can combine the above 12 equations, between initial (before impact) and final (after impact). let 'i' denote initial and 'f' denote final.

We have:

$(1) \Rightarrow r \int F_{sy}\,dt = I w_{xf} - I w_{xi}$

$(2) \Rightarrow -r \int F_{sx}\,dt = I w_{yf} - I w_{yi}$

$(3) \Rightarrow w_{zi} = w_{zf}$

$(4) \Rightarrow \int F_{sx}\,dt = m V_{Gxf} - m V_{Gxi}$

$(5) \Rightarrow \int F_{sy}\,dt = m V_{Gyf} - m V_{Gyi}$

$(6) \Rightarrow \int N\,dt = m V_{Gzf} - m V_{Gzi}$

N is the normal force during the impact between initial and final

We also have:

$$(7) \quad e = \frac{\int R\,dt}{\int P\,dt} = \frac{-V_{Gzf}}{V_{Gzi}}$$

coefficient of restitution, which must be known for the ball and surface combination

Let's assume, for simplicity, that the friction between ball and surface is high enough so that the ball stops slipping on the surface during impact and rolls without slipping just before it leaves the surface. Then,

$$\left.\begin{aligned} V_{Gxf} &= r\,w_{yf} \\ V_{Gyf} &= -r\,w_{xf} \end{aligned}\right\}$$ sub. into equations (4), (5)

use $I = \frac{2}{5} m r^2$ for a sphere (solid)

Solve equations (1) - (5), (7) and we get

$$w_{zf} = w_{zi}$$

$$V_{Gzf} = -e \cdot V_{Gzi}$$

All initial (i) values are known

$$w_{xf} = \frac{2r\,w_{xi} - 5V_{Gyi}}{7r}$$

$$w_{yf} = \frac{2r\,w_{yi} + 5V_{Gxi}}{7r}$$

$$V_{Gxf} = \frac{2r\,w_{yi} + 5V_{Gxi}}{7}$$

$$V_{Gyf} = \frac{-2r\,w_{xi} + 5V_{Gyi}}{7}$$

We must now revert back to the ground / observer frame. In this frame we see the surface moving at velocity $\vec{V}_s = V_{sx}\hat{I} + V_{sy}\hat{J} + V_{sz}\hat{K}$.

Hence, in the above expressions $\vec{V}_{Gi}$ must be replaced with $\vec{V}_{Gi}^{*} - \vec{V}_s$. Thus,

replace V_{Gxi} with $V_{Gxi}^{*} - V_{sx}$

replace V_{Gyi} with $V_{Gyi}^{*} - V_{sy}$

replace V_{Gzi} with $V_{Gzi}^{*} - V_{sz}$

and add V_{sx}, V_{sy}, V_{sz} to $\underbrace{V_{Gxf}, V_{Gyf}, V_{Gzf}}_{\text{(on previous page)}}$ resp.

Therefore,
the angular velocity and linear velocity $\underbrace{\text{after impact}}_{\text{immediately}}$ is:

FINAL AFTER IMPACT

$$\vec{w}_f = \left(\frac{2r\,w_{xi} - 5(V_{Gyi}^{*} - V_{sy})}{7r}\right)\hat{I} + \left(\frac{2r\,w_{yi} + 5(V_{Gxi}^{*} - V_{sx})}{7r}\right)\hat{J} + (w_{zi})\hat{K}$$

$$\vec{V}_{Gf} = \left(\frac{2r\,w_{yi} + 5V_{Gxi}^{*} + 2V_{sx}}{7}\right)\hat{I} + \left(\frac{-2r\,w_{xi} + 5V_{Gyi}^{*} + 2V_{sy}}{7}\right)\hat{J} + (-e\cdot V_{Gzi}^{*} + (1+e)V_{sz})\hat{K}$$