Spherical ball colliding with moving surface

Set up problem as follows:

- \( \vec{\omega}_i \) is the angular velocity of the ball just before it strikes the surface
- \( \vec{V}_{G,i}^* \) is the velocity of ball (at point \( G \)) just before it strikes the surface
- \( \vec{V}_s \) is the velocity of the surface before the ball strikes, at the point of impact
- \( r \) is the sphere radius

Assumptions:

- The moving surface remains at a constant velocity during the brief impact duration, the surface is rigid
- Gravity has a negligible effect during the brief impact duration, in which impulse forces dominate
- The spherical ball can be treated as a rigid body with negligible deformation during impact

Now, \( \vec{V}_{G,i}^* = \vec{V}_{ex,i} \hat{i} + \vec{V}_{ey,i} \hat{j} + \vec{V}_{ez,i} \hat{k} \)
\( \vec{w}_i = w_{x_i} \hat{i} + w_{y_i} \hat{j} + w_{z_i} \hat{k} \)  \\
\( \vec{v}_s = v_{sx} \hat{i} + v_{sy} \hat{j} + v_{sz} \hat{k} \)

\( \vec{w}_i, \vec{v}_i, \vec{v}_s \) are with respect to ground, which is the reference frame we, as observers, are in.

For convenience, set the reference frame so that it moves at constant velocity \( \vec{v}_s \). Then, relative to this reference frame the surface is stationary.

This problem is then transformed into a simpler one:

Where:

\[ \vec{v}_{ci} = \vec{v}_{ci}^* - \vec{v}_s \]

- \( x'y'z' \) is parallel to \( XYZ \)
- \( x'y'z' \) components are the same as \( XYZ \) components at \( G \)
- \( v_{ci} \) is the velocity of ball (at point \( G \)) just before it strikes the surface. This velocity is relative to the reference frame moving at velocity \( \vec{v}_s \)

Now, \( \vec{v}_{ci} = v_{cx_i} \hat{i} + v_{cy_i} \hat{j} + v_{cz_i} \hat{k} \)
Break the problem into two stages from initial, just before impact, to point at which \( V_{Gz} = 0 \). Apply impulse and momentum equations; given below.

\[
\begin{align*}
\vec{F} &= -r \hat{k} \\
\vec{M}_G &= -r \hat{k} \times (P \hat{k} + F_{sx} \hat{i}) \\
&\quad + F_{sy} \hat{j} \\
\end{align*}
\]

(moment about \( G \))

Where \( F_s \) is the friction force along the surface, with components \( F_{sx} \hat{i} + F_{sy} \hat{j} \), and \( P \) is the normal force along \( z \)-direction.

Now,

\[
\begin{align*}
\vec{M}_G &= -r F_{sx} \hat{j} + r F_{sy} \hat{i} \\
M_{xG} &= M_{yG} &= M_{zG} = 0 \\
\end{align*}
\]

\[
\begin{align*}
\int M_{xG} \, dt &= I w_{x2} - I w_{x1} \quad (1) \\
\int M_{yG} \, dt &= I w_{y2} - I w_{y1} \quad (2) \\
\int M_{zG} \, dt &= I w_{z2} - I w_{z1} \quad (3) \\
\int F_{sx} \, dt &= m V_{sx2} - m V_{sx1} \quad (4) \\
\int F_{sy} \, dt &= m V_{sy2} - m V_{sy1} \quad (5) \\
\int P \, dt &= m (0) - m V_{gz1} \quad (6) \\
\end{align*}
\]

where \( I \) is the principal moment of inertia of the ball about \( G \), and \( m \) is the ball mass.
Hence,

(1) \[ r \int F_{sy} \, dt = Iw_{x_2} - Iw_{x_1} \]
(2) \[ -r \int F_{sx} \, dt = Iw_{y_2} - Iw_{y_1} \]
(3) \[ 0 = Iw_{z_2} - Iw_{z_1} \Rightarrow w_{z_1} = w_{z_2} \]
(4) \[ \int F_{sx} \, dt = m V_{e_x}x_2 - m V_{e_x}x_1 \]
(5) \[ \int F_{sy} \, dt = m V_{e_y}x_2 - m V_{e_y}x_1 \]
(6) \[ SP \, dt = -m V_{e_z} \]

The second (final) stage of impact is from the point at which \( V_{e_z} = 0 \) to the point immediately after the ball has left the surface. The above equations for this stage become:

(1) \[ r \int F_{sy} \, dt = Iw_{x_3} - Iw_{x_2} \]
(2) \[ -r \int F_{sx} \, dt = Iw_{y_3} - Iw_{y_2} \]
(3) \[ w_{z_2} = w_{z_3} \]
(4) \[ \int F_{sx} \, dt = m V_{e_x}x_3 - m V_{e_x}x_2 \]
(5) \[ \int F_{sy} \, dt = m V_{e_y}x_3 - m V_{e_y}x_2 \]
(6) \[ SR \, dt = m V_{e_z} - 0 = m V_{e_z} \]

Replace \( P \) in the above equation (6) with \( R \).
We can combine the above 12 equations, between initial (before impact) and final (after impact). Let ‘i’ denote initial and ‘f’ denote final.

We have:

1. \( r \int F_x \, dt = I_w x_f - I_w x_i \)
2. \( -r \int F_y \, dt = I_w y_f - I_w y_i \)
3. \( w_{z_i} = w_{z_f} \)
4. \( \int F_x \, dt = m v_{x_f} - m v_{x_i} \)
5. \( \int F_y \, dt = m v_{y_f} - m v_{y_i} \)
6. \( \int N \, dt = m v_{z_f} - m v_{z_i} \)

\( N \) is the normal force during the impact between initial and final.

We also have:

7. \( e = \frac{SR dt}{\int S \rho dt} = \frac{v_{z_f}}{v_{z_i}} \)

Coefficient of restitution, which must be known for the ball and surface combination.
Let's assume, for simplicity, that the friction between ball and surface is high enough so that the ball stops slipping on the surface during impact and rolls without slipping just before it leaves the surface. Then,

\[ v_{x,f} = -w_{y,f} \]
\[ v_{y,f} = -w_{x,f} \]

\[ \{ \text{sub. m to equations (4), (5)} \}
\[ \text{use } I = \frac{2}{5}mr^2 \text{ for a sphere (solid)} \]

Solve equations (1) - (5), (7) and we get

\[ w_{zf} = w_{zi} \]
\[ w_{xf} = \frac{-e \cdot v_{yf}}{2r} \]
\[ w_{yp} = \frac{2r w_{yi} + 5v_{x}v_{yi}}{2r} \]
\[ v_{xf} = \frac{2r w_{yi} + 5v_{x}v_{yi}}{7} \]
\[ v_{yf} = \frac{-2r w_{yi} + 5v_{x}v_{yi}}{7} \]

All initial (i) values are known
We must now revert back to the ground/observer frame. In this frame we see the surface moving at velocity \( \vec{V}_s = V_{sx} \hat{i} + V_{sy} \hat{j} + V_{sz} \hat{k} \).

Hence, in the above expressions \( \vec{V} \) must be replaced with \( V_{\theta} \). Thus,

replace \( V_{gx} \) with \( V_{gx} - V_{sx} \)

replace \( V_{gy} \) with \( V_{gy} - V_{sy} \)

replace \( V_{gz} \) with \( V_{gz} - V_{sz} \)

and add \( V_{sx}, V_{sy}, V_{sz} \) to \( V_{ex}, V_{ey}, V_{ez} \) resp.

Therefore, the angular velocity and linear velocity after impact is:

\[
\vec{\omega}_f = \left( \frac{2 \tau \omega_{xi} - 5 (V_{gy} - V_{sy})}{7 \tau} \right) \hat{i} + \left( \frac{2 \tau \omega_{yi} + 5 (V_{gx} - V_{sx})}{7 \tau} \right) \hat{j} + (\omega_{zi}) \hat{k}
\]

\[
V_{g_f} = \left( \frac{2 \tau \omega_{yi} + 5 V_{gx} - 2 V_{sx}}{7 \tau} \right) \hat{i} + \left( -2 \tau \omega_{xi} + 5 V_{gy} + 2 V_{sy} \right) \hat{j} + (-e \cdot V_{gz} + (1+e) V_{sz}) \hat{k}
\]